מד"ר 1 / תרגיל בית #5

אריאל סטולרמן

(1)

בכל הסעיפים $f,g:U\to\mathbb{R}$ פונקציות חלקות, u,v שדות וקטוריים חלקים.

$a.\ L_v(f+g)=L_vf+L_vg$:

$$L_v(f+G)=\sum_{i=1}^{n}\frac{\partial(f+g)}{\partial x_i}v_i=\sum_{i=1}^{n}\left[\frac{\partial f}{\partial x_i}+\frac{\partial g}{\partial x_i}\right]v_i=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}v_i+\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}v_i=L_vf+L_vg$$

$b.\ L_v(fg)=fL_vg+gL_vf$:

$$L_v(fg)=\sum_{i=1}^{n}\frac{\partial(fg)}{\partial x_i}v_i=\sum_{i=1}^{n}\left[f\cdot\frac{\partial g}{\partial x_i}+g\cdot\frac{\partial f}{\partial x_i}\right]v_i=\sum_{i=1}^{n}f\cdot\frac{\partial g}{\partial x_i}v_i+\sum_{i=1}^{n}g\cdot\frac{\partial f}{\partial x_i}v_i=fL_vg+gL_vf$$

$c.\ L_{u+v}=L_u+L_v$:

$$\forall f:\ L_{u+v}f=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(u+v)_i=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(u_i+v_i)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}u_i+\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}v_i=L_uf+L_vf\Rightarrow L_{u+v}=L_u+L_v$$

$d.\ L_{fv}=fL_v$:

$$\forall g:\ L_{fv}g=\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(fv)_i=\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}f\cdot v_i=f\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}v_i=fL_vg\Rightarrow L_{fv}=fL_v$$

(2)

$L_uL_v\neq L_vL_u$:

$\forall f:\ L_uL_v=\sum_{i=0}^{n}\frac{\partial f}{\partial x_i}u_i\cdot\sum_{j=0}^{n}\frac{\partial f}{\partial x_j}v_j$ =???

(3)

יהיו $f,g:U\to\mathbb{R}$ אינטגרלים ראשונים למערכת משוואות $\acute{x}=v(x)$, כאשר v שדרה וקטורי:

$a.\ L_v(f+g)=0$:

$L_v(f+g)\underset{1.a\ \text{שאלה}}{=}L_vf+L_vg=0+0=0\Rightarrow$ אינטגרל ראשון $f+g$

$b.\ L_v(fg)=0$:

$L_v(fg)\underset{1.b\ \text{שאלה}}{=}fL_vg+gL_vf=f\cdot0+g\cdot0=0\Rightarrow$ אינטגרל ראשון fg

(4)

(5)

$a.\ A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$:

$$e^{At}=\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}=E+At+\frac{1}{2}\underbrace{A^2}_{=0}t^2+\cdots=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t\\0&0\end{pmatrix}=\begin{pmatrix}1&t\\0&1\end{pmatrix}$$

$b.\ A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$:

- $A^1=A$
- $A^2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-E$
- $A^3=-E\cdot A=-A$
- $A^4=E$

$$e^{At}=\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}=\sum_{k=0}^{\infty}\frac{(-1)^kAt^{2k+1}}{(2k+1)!}+\sum_{k=0}^{\infty}\frac{(-1)^kEt^{2k}}{(2k)!}=A\cdot sint+E\cdot cost=\begin{pmatrix}0&sint\\-sint&0\end{pmatrix}+\begin{pmatrix}cost&0\\0&cost\end{pmatrix}=\begin{pmatrix}cost&sint\\-sint&cost\end{pmatrix}$$

$c.\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

- $A^1 = A$
- $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$
- $A^3 = E \cdot A = A$
- $A^4 = E \cdot E = E$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} A \cdot \frac{t^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} E \cdot \frac{t^{2k}}{(2k)!} = A \cdot sinh(t) + E \cdot cosh(t) = \begin{pmatrix} cosh\,t & sinh\,t \\ sinh\,t & cosh\,t \end{pmatrix}$$

$d.\ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$:

$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^3 = \Theta$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = E + At + \frac{A^2 t^2}{2} + 0 + \cdots = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

(6)

$a.\ A = \begin{pmatrix} -1 & 5 \\ -10 & 14 \end{pmatrix}$:

$$p(A) = \begin{vmatrix} -1-\lambda & 5 \\ -10 & 14-\lambda \end{vmatrix} = (\lambda+1)(\lambda-14) + 50 = \lambda^2 - 13\lambda + 36 = 0 \Leftrightarrow \lambda_{1,2} = \frac{13 \pm \sqrt{169-144}}{2} \Rightarrow \lambda_1 = 9, \lambda_2 = 4 \Rightarrow$$

$D = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}; \quad D^k = \begin{pmatrix} 9^k & 0 \\ 0 & 4^k \end{pmatrix}$

הערכים העצמיים של A הם 9,4. נמצא ו״ע מתאימים:

$\lambda_1 = 9$: $\begin{pmatrix} -10 & 5 \\ -10 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -10x+5y \\ -10x+5y \end{pmatrix} = 0 \Rightarrow -10x + 5y = 0 \Rightarrow y = 2x \Rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = 4$: $\begin{pmatrix} -5 & 5 \\ -10 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -5x+5y \\ -10x+10y \end{pmatrix} = 0 \Rightarrow -5x + 5y = 0 \Rightarrow y = x \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{det\,P}\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \Rightarrow$

$A = P \cdot D \cdot P^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \Rightarrow e^{At} = P \cdot e^{Dt} \cdot P^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{9t} & 0 \\ 0 & e^{4t} \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} =$

$\begin{pmatrix} e^{9t} & e^{4t} \\ 2e^{9t} & e^{4t} \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2e^{4t} - e^{9t} & e^{9t} - e^{4t} \\ 2e^{4t} - 2e^{9t} & 2e^{9t} - e^{4t} \end{pmatrix}$

$b.\ A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$:

$$p(A) = \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda)(-4-\lambda) - 6 \cdot 2 \cdot 3 - 6 \cdot (-1) \cdot (-6) + 6(4-\lambda) \cdot 3 + 6 \cdot (-1)(-4-\lambda) -$$

$$-(5-\lambda) \cdot 2 \cdot (-6) = (\lambda-5)(16-\lambda^2) - 36 - 36 + 18(4-\lambda) + 6(4+\lambda) + 12(5-\lambda) = 16\lambda - \lambda^3 - 80 + 5\lambda^2 - 72 + 72 -$$

$$-18\lambda + 24 + 6\lambda + 60 - 12\lambda = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -\lambda^3 + \lambda^2 + 4\lambda^2 - 4\lambda - 4\lambda + 4 = (-\lambda^2 + 4\lambda - 4)(\lambda - 1) =$$

$$= -(\lambda-2)^2(\lambda-1) = 0 \Rightarrow \lambda_1 = 2\ (r_1 = 2), \lambda_2 = 1\ (r_2 = 1) \Rightarrow D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix}3&-6&-6\\-1&2&2\\3&-6&-6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow x=2y+2z\Rightarrow sp\{v_1+v_2\}=sp\left\{\begin{pmatrix}2t+2s\\t\\s\end{pmatrix}\right\}=sp\left\{t\begin{pmatrix}2\\1\\0\end{pmatrix}+s\begin{pmatrix}2\\0\\1\end{pmatrix}\right\}\Rightarrow v_1=\begin{pmatrix}2\\1\\0\end{pmatrix},v_2=\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}4&-6&-6\\-1&3&2\\3&-6&-5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow\begin{cases}2x=3y+3z\\x=3y+2z\\3x=6y+5z\end{cases}\Rightarrow\begin{cases}x=z\\z=-3y\\-9y=6y-15y\end{cases}\Rightarrow y=t\Rightarrow x=z=-3t\Rightarrow v_3=\begin{pmatrix}3\\-1\\3\end{pmatrix}$$

$$\Rightarrow P=\begin{pmatrix}2&2&3\\1&0&-1\\0&1&3\end{pmatrix}\Rightarrow\left(\begin{matrix}2&2&3\\1&0&-1\\0&1&3\end{matrix}\middle\|\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right)\to\left(\begin{matrix}1&2&4\\1&0&-1\\0&1&3\end{matrix}\middle\|\begin{matrix}1&-1&0\\0&1&0\\0&0&1\end{matrix}\right)\to\left(\begin{matrix}1&2&4\\0&-2&-5\\0&1&3\end{matrix}\middle\|\begin{matrix}1&-1&0\\-1&2&0\\0&0&1\end{matrix}\right)\to$$

$$\left(\begin{matrix}1&2&4\\0&1&4\\0&1&3\end{matrix}\middle\|\begin{matrix}1&-1&0\\-1&2&3\\0&0&1\end{matrix}\right)\to\left(\begin{matrix}1&2&4\\0&1&4\\0&0&-1\end{matrix}\middle\|\begin{matrix}1&-1&0\\-1&2&3\\1&-2&-2\end{matrix}\right)\to\left(\begin{matrix}1&2&4\\0&1&4\\0&0&1\end{matrix}\middle\|\begin{matrix}1&-1&0\\-1&2&3\\-1&2&2\end{matrix}\right)\to\left(\begin{matrix}1&2&0\\0&1&0\\0&0&1\end{matrix}\middle\|\begin{matrix}5&-9&-8\\3&-6&-5\\-1&2&2\end{matrix}\right)\to$$

$$\left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\middle\|\begin{matrix}-1&3&2\\3&-6&-5\\-1&2&2\end{matrix}\right)\Rightarrow P^{-1}=\begin{pmatrix}-1&3&2\\3&-6&-5\\-1&2&2\end{pmatrix}$$

$$e^{At}=P\cdot e^{Dt}\cdot P^{-1}=\begin{pmatrix}2&2&3\\1&0&-1\\0&1&3\end{pmatrix}\begin{pmatrix}e^{2t}&0&0\\0&e^{2t}&0\\0&0&e^t\end{pmatrix}\begin{pmatrix}-1&3&2\\3&-6&-5\\-1&2&2\end{pmatrix}=\begin{pmatrix}2e^{2t}&2e^{2t}&3e^t\\e^{2t}&0&-e^t\\0&e^{2t}&3e^t\end{pmatrix}\begin{pmatrix}-1&3&2\\3&-6&-5\\-1&2&2\end{pmatrix}=$$

$$\begin{pmatrix}-2e^{2t}+6e^{2t}-3e^t&6e^{2t}-12e^{2t}+6e^t&4e^{2t}-10e^{2t}+6e^t\\-e^{2t}+e^t&3e^{2t}-2e^t&2e^{2t}-2e^t\\3e^{2t}-3e^t&-6e^{2t}+6e^t&-5e^{2t}+6e^t\end{pmatrix}=\begin{pmatrix}4e^{2t}-3e^t&-6e^{2t}+6e^t&-6e^{2t}+6e^t\\-e^{2t}+e^t&3e^{2t}-2e^t&2e^{2t}-2e^t\\3e^{2t}-3e^t&-6e^{2t}+6e^t&-5e^{2t}+6e^t\end{pmatrix}$$

(7)

$a.\begin{cases}x_1'=4x_1-3x_2\\x_2'=8x_1-6x_2\end{cases}\Rightarrow A=\begin{pmatrix}4&-3\\8&-6\end{pmatrix}$:

$$p(A)=\begin{vmatrix}4-\lambda&-3\\8&-6-\lambda\end{vmatrix}=-24-4\lambda+6\lambda+\lambda^2+24=\lambda^2+2\lambda=0\Rightarrow\lambda_1=0,\lambda_2=-2\Rightarrow D=\begin{pmatrix}0&0\\0&-2\end{pmatrix}$$

$$\lambda_1=0:\ \begin{pmatrix}4&-3\\8&-6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0\Rightarrow 4x=3y\Rightarrow x=\frac{3}{4}y\Rightarrow v_1=\begin{pmatrix}3\\4\end{pmatrix};\quad\lambda_2=-2:\ \begin{pmatrix}6&-3\\8&-4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0\Rightarrow 2x=y\Rightarrow v_2=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\Rightarrow P=\begin{pmatrix}3&1\\4&2\end{pmatrix}\Rightarrow P^{-1}=\frac{1}{2}\begin{pmatrix}2&-1\\-4&3\end{pmatrix}=\begin{pmatrix}1&-0.5\\-2&1.5\end{pmatrix}\Rightarrow\quad Solution:\left\{\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}e^{-2t}\right\}\Rightarrow x(t)=c_1\begin{pmatrix}3\\4\end{pmatrix}+c_2\begin{pmatrix}1\\2\end{pmatrix}e^{-2t}$$

$b.\begin{cases}x_1'=x_1+x_2+x_3\\x_2'=2x_1+x_2-x_3\\x_3'=-8x_1-5x_2-3x_3\end{cases}\Rightarrow A=\begin{pmatrix}1&1&1\\2&1&-1\\-8&-5&-3\end{pmatrix}$:

$$p(A)=\begin{vmatrix}1-\lambda&1&1\\2&1-\lambda&-1\\-8&-5&-3-\lambda\end{vmatrix}=(1-\lambda)(1-\lambda)(-3-\lambda)+8-10+8(1-\lambda)+2(3+\lambda)-5(1-\lambda)=$$

$$=(1-\lambda)(-3-\lambda+3\lambda+\lambda^2+8-5)-2+6+2\lambda=(1-\lambda)(\lambda^2+2\lambda)+4+2\lambda=\lambda(1-\lambda)(\lambda+2)+2(\lambda+2)$$

$$=(\lambda+2)(\lambda[1-\lambda]+2)=(\lambda+2)(-\lambda^2+\lambda+2)=-(\lambda+2)(\lambda^2-\lambda-2)=-(\lambda+2)(\lambda+1)(\lambda-2)\Rightarrow$$

$\lambda_1=2,\lambda_2=-2,\lambda_3=-1$

$$\begin{pmatrix}-1&1&1\\2&-1&-1\\-8&-5&-5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0:\ \begin{pmatrix}-1&1&1\\2&-1&-1\\-8&-5&-5\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}\Rightarrow x=0,y=-z\Rightarrow v_1=\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}3&1&1\\2&3&-1\\-8&-5&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0:\ \begin{pmatrix}3&1&1\\2&3&-1\\-8&-5&-1\end{pmatrix}\to\begin{pmatrix}7&0&4\\0&7&-5\\0&0&0\end{pmatrix}\Rightarrow x=-\frac{4}{7}z,y=\frac{5}{7}z\Rightarrow v_2=\begin{pmatrix}-4\\5\\7\end{pmatrix}$$

$$\begin{pmatrix}2&1&1\\2&2&-1\\-8&-5&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0:\ \begin{pmatrix}2&1&1\\2&2&-1\\-8&-5&-2\end{pmatrix}\to\begin{pmatrix}2&0&3\\0&1&-2\\0&0&0\end{pmatrix}\Rightarrow x=-\frac{3}{2}z,y=2z\Rightarrow v_3=\begin{pmatrix}-3\\4\\2\end{pmatrix}$$

$$\Rightarrow Solution:\ x(t)=c_1\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}-4\\5\\7\end{pmatrix}e^{-2t}+c_3\begin{pmatrix}-3\\4\\2\end{pmatrix}e^{-t}$$

$c.\begin{cases}x_1' = x_1 - 5x_2\\ x_2' = x_1 - 3x_2\end{cases} \Rightarrow A = \begin{pmatrix}1 & -5\\ 1 & -3\end{pmatrix}$:

$$p(A) = \begin{vmatrix}1-\lambda & -5\\ 1 & -3-\lambda\end{vmatrix} = (1-\lambda)(-3-\lambda)+5 = -3-\lambda+3\lambda+\lambda^2+5 = \lambda^2+2\lambda+2 \Rightarrow$$

$$\lambda_{1,2} = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i \Rightarrow$$

$$\begin{pmatrix}2+i & -5\\ 1 & -2+i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}2x-5y+ix\\ x-2y+iy\end{pmatrix} = 0 \Rightarrow ???$$

$d.\begin{cases}x_1' = 4x_1 - 2x_2\\ x_2' = 8x_1 - 4x_2\end{cases} \Rightarrow A = \begin{pmatrix}4 & -2\\ 8 & -4\end{pmatrix}$:

$$p(A) = \begin{vmatrix}4-\lambda & -2\\ 8 & -4-\lambda\end{vmatrix} = -(16-\lambda^2)+16 = \lambda^2 \Rightarrow \lambda = 0\ (r=2) \Rightarrow$$

$$\begin{pmatrix}4 & -2\\ 8 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}4x-2y\\ 8x-4y\end{pmatrix} = 0 \Rightarrow 2x = y \Rightarrow v = \begin{pmatrix}1\\ 2\end{pmatrix} \Rightarrow \ \textit{first solution}: \begin{pmatrix}1\\ 2\end{pmatrix}$$

$\textit{second solution}$: $x(t) = (\psi_1+\psi_2 t)e^0 = \psi_1 + \psi_2 t;\ \ \psi_1 = \begin{pmatrix}a\\ b\end{pmatrix}, \psi_2 = \begin{pmatrix}c\\ d\end{pmatrix} \Rightarrow$

$$x(t) = \begin{pmatrix}a\\ b\end{pmatrix} + \begin{pmatrix}c\\ d\end{pmatrix}t = \begin{pmatrix}a+ct\\ b+dt\end{pmatrix};\ \ x'(t) = \begin{pmatrix}c\\ d\end{pmatrix};\ \ x' = Ax \Rightarrow \begin{pmatrix}c\\ d\end{pmatrix} = \begin{pmatrix}4 & -2\\ 8 & -4\end{pmatrix}\begin{pmatrix}a+ct\\ b+dt\end{pmatrix} = \begin{pmatrix}4a+4ct-2b-2dt\\ 8a+8ct-4b-4dt\end{pmatrix} \Rightarrow$$

$$\begin{cases}c = 4a-2b+(4c-2d)t\\ d = 8a-4b+(8c-4d)t\end{cases} \Rightarrow \begin{cases}4c-2d=0\\ 8c-4d=0\\ 4a-2b=c\\ 8a-4b=d\end{cases} \Rightarrow \begin{cases}d = 2c\\ b = 2a-\dfrac{c}{2}\end{cases} \Rightarrow c \coloneqq 4, a \coloneqq 1 \Rightarrow x(t) = \begin{pmatrix}1\\ 0\end{pmatrix} + \begin{pmatrix}4\\ 8\end{pmatrix}t \Rightarrow$$

$\textit{solution}$: $x(t) = c_1\begin{pmatrix}1\\ 2\end{pmatrix} + c_2\left[\begin{pmatrix}1\\ 0\end{pmatrix} + \begin{pmatrix}4\\ 8\end{pmatrix}t\right]$

$e.\begin{cases}x_1' = x_1+x_2+x_3\\ x_2' = 2x_1+x_2-x_3\\ x_3' = -3x_1+2x_2+4x_3\end{cases} \Rightarrow A = \begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ -3 & 2 & 4\end{pmatrix}$:

$$p(A) = \begin{vmatrix}1-\lambda & 1 & 1\\ 2 & 1-\lambda & -1\\ -3 & 2 & 4-\lambda\end{vmatrix} = (1-\lambda)^2(4-\lambda)+3+4+3(1-\lambda)-2(4-\lambda)+2(1-\lambda) = (1-2\lambda+\lambda^2)(4-\lambda)+7+$$

$$3-3\lambda-8+2\lambda+2-2\lambda = 4-8\lambda+4\lambda^2-\lambda+2\lambda^2-\lambda^3+4-3\lambda = -\lambda^3+6\lambda^2-12\lambda+8 = -\lambda^3+2\lambda^2+4\lambda^2-8\lambda-4\lambda+8$$

$$= -\lambda^2(\lambda-2)+4\lambda(\lambda-2)-4(\lambda-2) = (\lambda-2)(-\lambda^2+4\lambda-4) = -(\lambda-2)(\lambda-2)^2 = -(\lambda-2)^3 \Rightarrow \lambda = 2\ \ (r=3)$$

$$\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}-x+y+z\\ 2x-y-z\\ -3x+2y+2z\end{pmatrix} = 0 \Rightarrow \begin{cases}x = y+z\\ 2x = y+z \Rightarrow x = 0, y = -z \Rightarrow v = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix}\\ 3x = 2y+2z \Rightarrow y = -z\end{cases}$$

$$\Rightarrow \textit{first solution}: \begin{pmatrix}0\\ 1\\ -1\end{pmatrix}e^{2t}$$

$\textit{second and third solutions}$: $(\psi_1+\psi_2 t)e^{2t}, (\varphi_1+\varphi_2 t+\varphi_3 t^2)e^{2t}$

$$\psi_1 = \begin{pmatrix}a\\ b\\ c\end{pmatrix}, \psi_2 = \begin{pmatrix}d\\ e\\ f\end{pmatrix} \Rightarrow x(t) = \begin{pmatrix}a\\ b\\ c\end{pmatrix}e^{2t} + \begin{pmatrix}d\\ e\\ f\end{pmatrix}te^{2t}, x'(t) = \begin{pmatrix}2a\\ 2b\\ 2c\end{pmatrix}e^{2t} + \begin{pmatrix}d\\ e\\ f\end{pmatrix}e^{2t} + \begin{pmatrix}2dt\\ 2et\\ 2ft\end{pmatrix}e^{2t}, x' = Ax \Rightarrow$$

$$\begin{pmatrix}2a+d+2dt\\ 2b+e+2et\\ 2c+f+2ft\end{pmatrix}e^{2t} = \begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ -3 & 2 & 4\end{pmatrix}\begin{pmatrix}a+dt\\ b+et\\ c+ft\end{pmatrix}e^{2t} = \begin{pmatrix}a+b+c+(d+e+f)t\\ 2a+b-c+(2d+e-f)t\\ -3a+2b+4c+(-3d+2e+4f)t\end{pmatrix}e^{2t} \Rightarrow$$

$$\begin{cases}2a+d = a+b+c\\ 2b+e = 2a+b-c\\ 2c+f = -3a+2b+4c\\ 2d = d+e+f\\ 2e = 2d+e-f\\ 2f = -3d+2e+4f\end{cases} \Rightarrow \begin{cases}a = b+c-d\\ b = 2a-c-e\\ -2c = -3a+2b-f\\ d-e-f = 0\\ -2d+e+f = 0\\ 3d-2e-2f = 0\end{cases};\ \ \begin{pmatrix}1 & -1 & -1\\ -2 & 1 & 1\\ 3 & -2 & -2\end{pmatrix} \to \begin{pmatrix}1 & -1 & -1\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \Rightarrow d = 0, e = -f; \textit{let } e \coloneqq 1 \Rightarrow$$

$$\begin{cases} a = b + c \\ b = 2a - c - 1 \\ -2c = -3a + 2b + 1 \end{cases} \Rightarrow \begin{cases} b = 2b + 2c - c - 1 \\ -2c = -3b - 3c + 2b + 1 \end{cases} \Rightarrow \begin{cases} b = 1 - c \\ \underbrace{-2c = -3 + 3c - 3c + 2 - 2c + 1}_{0=0} \end{cases} \Rightarrow let\ c \coloneqq 1 \Rightarrow a = 1, b = 0 \Rightarrow$$

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \psi_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow second\ solution: x(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t}$$

$$\varphi_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \varphi_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}, \varphi_3 = \begin{pmatrix} g \\ h \\ i \end{pmatrix} \Rightarrow x(t) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} e^{2t} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} te^{2t} + \begin{pmatrix} g \\ h \\ i \end{pmatrix} t^2 e^{2t},$$

$$x'(t) = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} e^{2t} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} e^{2t} + \begin{pmatrix} 2dt \\ 2et \\ 2ft \end{pmatrix} e^{2t} + \begin{pmatrix} 2gt \\ 2ht \\ 2it \end{pmatrix} e^{2t} + \begin{pmatrix} 2gt^2 \\ 2ht^2 \\ 2it^2 \end{pmatrix} e^{2t}, \qquad x' = Ax \Rightarrow$$

$$\begin{pmatrix} 2a + d + (2d + 2g)t + 2gt^2 \\ 2b + e + (2e + 2h)t + 2ht^2 \\ 2c + f + (2f + 2i)t + 2it^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \begin{pmatrix} a + dt + gt^2 \\ b + et + ht^2 \\ c + ft + it^2 \end{pmatrix} =$$

$$\begin{pmatrix} a + b + c + (d + e + f)t + (g + h + i)t^2 \\ 2a + b - c + (2d + e - f)t + (2g + h - i)t^2 \\ -3a + 2b + 4c + (-3d + 2e + 4f)t + (-3g + 2h + 4i)t^2 \end{pmatrix} \Rightarrow \begin{cases} 2a + d = a + b + c \\ 2b + e = 2a + b - c \\ 2c + f = -3a + 2b + 4c \end{cases}, \begin{cases} 2d + 2g = d + e + f \\ 2e + 2h = 2d + e - f \\ 2f + 2i = -3d + 2e + 4f \end{cases},$$

$$\begin{cases} 2g = g + h + i \\ 2h = 2g + h - i \\ 2i = -3g + 2h + 4i \end{cases} \Rightarrow \begin{cases} -a + b + c - d = 0 \\ 2a - b - c - e = 0 \\ -3a + 2b + 2c - f = 0 \end{cases}, \begin{cases} -d + e + f - 2g = 0 \\ 2d - e - f - 2h = 0 \\ -3d + 2e + 2f - 2i = 0 \end{cases}, \begin{cases} g - h - i = 0 \\ 2g - h - i = 0 \\ -3g + 2h + 2i = 0 \end{cases} \Rightarrow$$

$$\begin{pmatrix} -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -3 & 2 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 & 2 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 2 & 2 \end{pmatrix} = 0 \Rightarrow \begin{cases} a = 2 \\ b = 2 \\ c = 2 \end{cases} \Rightarrow \begin{cases} d = 2 \\ e = 0 \\ f = 2 \\ g = 0 \\ h = 1 \\ i = -1 \end{cases} \Rightarrow \varphi_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \varphi_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \varphi_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow$$

$$third\ solution: x(t) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t^2 e^{2t}$$

$$\Rightarrow solution: x(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t \right] e^{2t} + c_3 \left[\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t^2 \right] e^{2t}$$

לא יצא כמו בפתרון אבל הרעיון ברור