מד"ר 1 / תרגיל בית 3#

אריאל סטולרמן

המשך תרגיל 2 – מציאת פיתוח $y(x)$ עד סדר 4:

1) $y' = x^2 + y^2, y(0) = 1$:

מתקיים: $x_0 = 0, c_0 = 1$.

$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \Rightarrow$

$y'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3$

$y'(x) = x^2 + y^2 = x^2 + (1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4)^2 \cong x^2 + 1 + c_1x + c_2x^2 + c_3x^3 + c_1x + c_1^2x^2 + c_1c_2x^3 + c_2x^2 + c_1c_2x^3$

$+c_3x^3 = 1 + 2c_1x + (1 + c_1^2 + 2c_2)x^2 + (2c_1c_2 + 2c_3)x^3 \Rightarrow$

$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 = 1 + 2c_1x + (1 + c_1^2 + 2c_2)x^2 + (2c_1c_2 + 2c_3)x^3 \Rightarrow$

$c_0 = 1; \quad c_1 = 1; \quad 2c_2 = 2c_1 = 2 \Rightarrow c_2 = 1; \quad 3c_3 = 1 + c_1^2 + 2c_2 = 1 + 1 + 2 = 4 \Rightarrow c_3 = \frac{4}{3}; \quad 4c_4 = 2c_1c_2 + 2c_3 = 4\frac{2}{3} \Rightarrow$

$y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + 4\frac{2}{3}x^4 + \cdots$

2) $y'' = e^y + x, y(0) = 1, y'(0) = 0$:

מתקיים: $x_0 = 0, c_0 = 1, c_1 = 0$:

$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 = 1 + c_2x^2 + c_3x^3 + c_4x^4 \Rightarrow$

$y''(x) = 2c_2 + 6c_3x + 12c_4x^2$

$e^y \cong e^{1+c_2x^2+c_3x^3+c_4x^4} = e \cdot e^{c_2x^2} \cdot e^{c_3x^3} \cdot e^{c_4x^4} \cong$

$$\left[e^{c_2x^2} = \sum_{k=0}^{\infty} \frac{(c_2x^2)^k}{k!} = 1 + c_2x^2 + o(x^2); \quad e^{c_3x^3} = 1 + o(x^2); \quad e^{c_4x^4} = 1 + o(x^2)\right]$$

$\cong e \cdot (1 + c_2x^2) \cdot 1 \cdot 1 = e + c_2ex^2 \Rightarrow$

$2c_2 + 6c_3x + 12c_4x^2 = e + x + c_2ex^2 \Rightarrow c_0 = 1,; \quad c_1 = 0; \quad 2c_2 = e \Rightarrow c_2 = \frac{e}{2}; \quad 6c_3 = 1 \Rightarrow c_3 = \frac{1}{6}; \quad 12c_4 = c_2e \Rightarrow c_4 = \frac{e^2}{24}$

$\Rightarrow y(x) = 1 + \frac{e}{2}x^2 + \frac{1}{6}x^3 + \frac{e^2}{24}x^4 + \cdots$

3) $y'' = yy' - x^2, y(0) = 1, y'(0) = 1$:

מתקיים: $x_0 = 0, c_0 = 1, c_1 = 1$.

$y(x) = \cdots = 1 + x + c_2x^2 + c_3x^3 + c_4x^4 \Rightarrow$

$y'(x) = 1 + 2c_2x + 3c_3x^2 + 4c_4x^3, \;\; y''(x) = 2c_2 + 6c_3x + 12c_4x^2$

$yy' = (1 + x + c_2x^2 + c_3x^3 + c_4x^4)(1 + 2c_2x + 3c_3x^2 + 4c_4x^3) \cong 1 + 2c_2x + 3c_3x^2 + x + 2c_2x^2 + c_2x^2 \Rightarrow$

$y'' = yy' - x^2 \Rightarrow 2c_2 + 6c_3x + 12c_4x^2 = 1 + (2c_2 + 1)x + (3c_2 + 3c_3 - 1)x^2 \Rightarrow$

$c_0 = 1; \quad c_1 = 1; \quad 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2}; \quad 6c_3 = 2c_2 + 1 \Rightarrow c_3 = \frac{2}{6}; \quad 12c_4 = 3c_2 + 3c_3 - 1 = \frac{3}{2} + 1 - 1 = \frac{3}{2} \Rightarrow c_4 = \frac{3}{24} \Rightarrow$

$y(x) = 1 + x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{3}{4!}x^4 + \cdots$

תרגיל 3 :

(1) $Ax = x^2, x_0 = \frac{1}{2}$:

- $x_0 = \frac{1}{2}$
- $x_1 = Ax_0 = \frac{1}{2^2}$
- $x_2 = Ax_1 = \frac{1}{(2^2)^2} = \frac{1}{2^4}$
- $x_3 = Ax_2 = \frac{1}{(2^4)^2} = \frac{1}{2^8}$

$\Rightarrow x_n = \frac{1}{2^{2^n}}$

(2) $Ax = \frac{1}{1+x}, x \in [0.5,1]$:

נקודות שבת:

$$x \in [0.5,1], Ax = x \Rightarrow \frac{1}{1+x} = x \Rightarrow 1 = x + x^2 \Rightarrow x^2 + x - 1 = 0 \Rightarrow x_{1,2} = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow x_1 = -\frac{1+\sqrt{5}}{2}, x_2 = \frac{-1+\sqrt{5}}{2}$$

נבדוק איזה מהתוצאות בתחום:

$$A\left(-\frac{1+\sqrt{5}}{2}\right) = \frac{1}{1 - \frac{1+\sqrt{5}}{2}} = \frac{2}{2-1-\sqrt{5}} = \frac{2}{1-\sqrt{5}} = -1.618 \notin [0.5,1]$$

$$A\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{1}{1 + \frac{-1+\sqrt{5}}{2}} = \frac{2}{2-1+\sqrt{5}} = \frac{2}{1+\sqrt{5}} = 0.618 \in [0.5,1]$$

מכאן שנקודת השבת היחידה בתחום היא: $x = \frac{2}{1+\sqrt{5}}$.

מרחב מטרי: $(M,\rho) = ([0.5,1], |x-y|)$. בדיקה האם A מכווצת:

$$\rho(Ax,Ay) = \left|\frac{1}{1+x} - \frac{1}{1+y}\right| = \left|\frac{y-x}{(x+1)(y+1)}\right| = \left|\frac{1}{(x+1)(y+1)}\right| \cdot |x-y| \le \max_{[0.5,1]}\left|\frac{1}{(x+1)(y+1)}\right| \cdot |x-y| = \frac{1}{1.5^2}|x-y|$$

$= \frac{4}{9}|x-y| \quad \Rightarrow \lambda = \frac{4}{9} < 1 \Rightarrow$ העתקה מכווצת A

(3) $A(x_1,x_2) = (x_1,-x_2)$:

תהי $x = (x_1,x_2)$ נקודת שבת של A:

$$Ax = x \Rightarrow A(x_1,x_2) = (x_1,-x_2) = (x_1,x_2) \qquad \Rightarrow x \in \{(\alpha,0) |\ \alpha \in \mathbb{R}\}$$

(4) $A(x_1,x_2) = (x_1,x_2^2)$:

$$Ax = x \Rightarrow A(x_1,x_2) = (x_1,x_2^2) = (x_1,x_2) \Rightarrow x_2 = x_2^2 \Rightarrow x_2 \in \{0,1\} \qquad \Rightarrow x \in \{(\alpha,\beta) \mid \alpha \in \mathbb{R}, \beta \in \{0,1\}\}$$

(5) $x' = 2tx^2, x(1) = 2, V = \{|t-1| \le 1, |x-2| \le 1\}$:

מתקיים: $t_0 = 1, x_0 = 2, a = 1, b = 1, f = 2tx^2$.

$$|t-1| \le 1 \Rightarrow -1 \le t-1 \le 1 \Rightarrow 0 \le t \le 2; \qquad |x-2| \le 1 \Rightarrow -1 \le x-2 \le 1 \Rightarrow 1 \le x \le 3 \quad \Rightarrow$$

$$c = \max_V|f| = \max_V|2tx^2| = 2 \cdot 2 \cdot 3^2 = 36$$

$$\delta = \min\left\{a, \frac{b}{c}\right\} = \min\left\{1, \frac{1}{36}\right\} = \frac{1}{36} \Rightarrow$$

By P-L theorem, the segment is: $[t_0 - \delta, t_0 + \delta] = \left[1 - \frac{1}{36}, 1 + \frac{1}{36}\right]$

(6) $x' = 2tx^2, x(0) = 1, V = \{|t| \le \frac{\sqrt{2}}{4}, |x-1| \le 1\}$:

מתקיים: $t_0 = 0, x_0 = 1, a = \frac{\sqrt{2}}{4}, b = 1, f = 2tx^2$:

$$|t| \le \frac{\sqrt{2}}{4} \Rightarrow -\frac{\sqrt{2}}{4} \le t \le \frac{\sqrt{2}}{4}; \qquad |x-1| \le 1 \Rightarrow -1 \le x-1 \le 1 \Rightarrow 0 \le x \le 2 \quad \Rightarrow$$

$$c = \max_V |2tx^2| = 2 \cdot \frac{\sqrt{2}}{4} \cdot 2^2 = 2\sqrt{2} = \frac{4}{\sqrt{2}}$$

$$\delta = \min\left\{a, \frac{b}{c}\right\} = \min\{\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\} = \frac{\sqrt{2}}{4} \Rightarrow$$

By P-L theorem, the segment is: $\left[-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right]$

(7) $x' + p(t)x = q(t)$:

כדי שלמשוואה יהיה פתרון יחיד, צריך שיתקיים ש-$f(t,x) = q(t) - p(t)x$ תהיה רציפה ב-$\mathbb{R}$ וש-$f_x' = p(t)$ תהיה רציפה ב-$\mathbb{R}$. כלומר, סה״כ, ש-$p(t), q(t)$ יהיו רציפות ב-$\mathbb{R}$.

(8) $x' = t + x, x(0) = 1$:

מתקיים: $x_0 = 1, t_0 = 0$.

$$\varphi_0 \equiv x_0 = 1$$
$$\varphi_1 = x_0 + \int_{t_0}^{t} f\left[\tau, \underbrace{\varphi_0(\tau)}_{=x_0=1}\right] d\tau = 1 + \int_0^t (\tau + 1) d\tau = 1 + t + \frac{t^2}{2}$$
$$\varphi_2 = x_0 + \int_0^t f[\tau, \underbrace{\varphi_1(\tau)}_{=1+\tau+\frac{\tau^2}{2}}] d\tau = 1 + \int_0^t \left(1 + 2\tau + \frac{\tau^2}{2}\right) d\tau = 1 + t + t^2 + \frac{t^3}{6}$$
$$\varphi_3 = x_0 + \int_0^t f[\tau, \underbrace{\varphi_2(\tau)}_{=1+\tau+\tau^2+\frac{\tau^3}{6}}] d\tau = 1 + \int_0^t \left(1 + 2\tau + \tau^2 + \frac{\tau^3}{6}\right) d\tau = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}$$
$$\varphi_4 = x_0 + \int_0^t f[\tau, \underbrace{\varphi_3(\tau)}_{=1+\tau+\cdots}] d\tau = 1 + \int_0^t \left(1 + 2\tau + \tau^2 + \frac{\tau^3}{3} + \frac{\tau^4}{24}\right) d\tau = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{120}$$
$$\varphi_5 = 1 + \int_0^t \left(1 + 2\tau + \tau^2 + \frac{\tau^3}{3} + \frac{\tau^4}{12} + \frac{\tau^5}{120}\right) d\tau = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{60} + \frac{t^6}{720}$$
$$\Rightarrow \varphi(t) = 2 + 2t + 2 \cdot \frac{t^2}{2} + 2 + \frac{t^3}{6} + 2 \cdot \frac{t^4}{24} + 2 \cdot \frac{t^5}{120} + \cdots - t - 1 = 2\sum_{k=0}^{\infty} \frac{t^k}{k!} - t - 1 = 2e^t - t - 1$$

הפתרון הנ״ל מוגדר לכל $t \in \mathbb{R}$.

(9) משוואת *Bessel*: $x^2y'' + xy' + (x^2 - p^2)y = 0$, כאשר $p = \frac{1}{2}$. בתרגול ראינו:

$$J_p(x) = x^p \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!\,(2p+2)\ldots(2p+2k)} x^{2k} \quad \Rightarrow$$

$$J_{\frac{1}{2}}(x) = \sqrt{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!\,(1+2)(1+4)\ldots(1+2k)} x^{2k} = \sqrt{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!\,(2k+1)!!} x^{2k} = \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \frac{1}{\sqrt{x}} sinx$$

$$J_{-\frac{1}{2}}(x) = x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!\,(2-1)(4-1)\ldots(2k-1)} x^{2k} = \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!\,(2k-1)!!} x^{2k} = \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \frac{1}{\sqrt{x}} cosx$$