Home Assignment III

1. Write down the iteration sequence for the mapping $Ax = x^2$ if $x_0 = \dfrac{1}{2}$.

2. Find all the fixed points of the mapping $Ax = \dfrac{1}{1+x}$ on the segment $[0.5, 1]$. Whether this mapping is contractive?

Find all fixed points of the mapping $A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$:

3. $A(x_1, x_2) = (x_1, -x_2)$;

4. $A(x_1, x_2) = (x_1, x_2^2)$.

Find a segment $[\alpha, \beta]$ for which the existence of solution is garanteed for the given initial value problem:

5. $x' = 2tx^2$, $x(1) = 2$, on the compact $V = \{|t-1| \leqslant 1, \quad |x-2| \leqslant 1\}$;

6. $x' = 2tx^2$, $x(0) = 1$, on the compact $V = \{|t| \leqslant \frac{\sqrt{2}}{4}, \quad |x-2| \leqslant 1\}$.

7. What conditions should be put on functions $p(t)$ and $q(t)$ s.t. the linear equation $x' + p(t)x = q(t)$ will have a unique solution for any initial condition $x_(t_0) = x_0$?

8. Using the Picard process, find the exact solution for the initial value problem $x' = t + x$, $x(0) = 1$. What is the domain for the obtained solution?

9. Denote by $J_p(x)$ the p-th Bessel function, i.e. the solution of the equation $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ corresponding to $r = \dfrac{1}{2}$. Check by direct computation that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

10. Formulate the Picard's Existence and Uniqueness Theorem for a system of ordinary differential equations of the first order. Modify the proof given in the class to fit this case. (This exercise is not compulsory, but it is very useful.)

Answers:

1. $x_n = \dfrac{1}{2^{2n}}$.
2. $X = \dfrac{2}{1+\sqrt{5}}$, A is a contraction.
3. The axis $x_2 = 0$.
4. The straight lines $x_2 = 0$ and $x_2 = 1$.
5. $[1 - \frac{1}{36}, 1 + \frac{1}{36}]$.
6. $[-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}]$.
7. Continuity of p and q on $\mathbb{R}$ is enough.
8. $x(t) = 2e^t - t - 1$, defined for all t's.

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