

מד"ר / תרגיל בית #2

אריאל סטורמן

שיטת אויילר:

$$: x' = \frac{x}{t}, x(1) = 1, t = 4, h = \frac{1}{2} \cdot 1$$

(א) מתקיים כי $f(t, x) = \frac{x}{t}$, $t_0 = 1, x_0 = 1$. נסמן $t_i = 1 + 0.5i$ עבור $0 \leq i \leq 6$ (כאשר $t = 4$) $t_6 = 1 + 0.5 \cdot 6 = 4$.

$$i. (t_0, x_0) = (1, 1) \Rightarrow m_0 = f(1, 1) = \frac{1}{1} = 1 \Rightarrow x - 1 = 1 \cdot (t - 1) \Rightarrow x = t \Rightarrow (t_1, x_1) = (1.5, 1.5)$$

$$ii. m_1 = f(1.5, 1.5) = 1 \Rightarrow x - 1.5 = 1 \cdot (t - 1.5) \Rightarrow x = t \Rightarrow (t_2, x_2) = (2, 2)$$

iii. ...

משוואת הישר המתקבלת בכל שלב הינה קבועה: $x = t$, ולפיכך בנקודה $t = 4$ נקבל $x(4) = 4$.

(ב) הפתרון המדויק למד"ר:

$$x' = \frac{t}{x} \Rightarrow \frac{dx}{dt} = \frac{t}{x} \Rightarrow \int x dx = \int t dt \Rightarrow \frac{x^2}{2} = \frac{t^2}{2} + \frac{c}{2} \Rightarrow x = \sqrt{t^2 + c}; \quad x(1) = 1 \Rightarrow 1 = \sqrt{1 + c} \Rightarrow c = 0 \Rightarrow x = t$$

פתרון זה זהה לפתרון המקורב לפי שיטת אויילר, וגם כאן $x(4) = 4$.

$$: x' = t^2 + x^2, x(0) = 0, t = 1, h = 0.2 \cdot 2$$

(א) מתקיים $f(t, x) = t^2 + x^2$, $t_0 = 0, x_0 = 0$. נסמן $t_i = 0.2i$ כאשר $0 \leq i \leq 5$ (כאשר $t = 1$) $t_5 = 0.2 \cdot 5 = 1$.

$$i. (t_0, x_0) = (0, 0) \Rightarrow m_0 = f(0, 0) = 0 \Rightarrow x - 0 = 0 \cdot (t - 0) \Rightarrow x \equiv 0 \Rightarrow (t_1, x_1) = (0.2, 0) \Rightarrow$$

$$ii. m_1 = f(0.2, 0) = 0.2^2 = 0.04 \Rightarrow x - 0 = 0.04 \cdot (t - 0.2) \Rightarrow x = 0.04t - 0.008 \Rightarrow (t_2, x_2) = (0.4, 0.008) \Rightarrow$$

$$iii. m_2 = f(0.4, 0.008) \cong 0.16 \Rightarrow x - 0.008 = 0.16(t - 0.4) \Rightarrow x = 0.16t - 0.056 \Rightarrow (t_3, x_3) = (0.6, 0.04) \Rightarrow$$

$$iv. m_3 = f(0.6, 0.04) \cong 0.3616 \Rightarrow x - 0.04 = 0.3616(t - 0.6) \Rightarrow x = 0.3616t - 0.1769 \Rightarrow (t_4, x_4) = (0.8, 0.1123) \Rightarrow$$

$$v. m_4 = f(0.8, 0.1123) \cong 0.6526 \Rightarrow x - 0.1123 = 0.6526(t - 0.8) \Rightarrow x = 0.6526t - 0.4097 \Rightarrow (t_5, x_5) = (1, 0.2428) \Rightarrow$$

ומכאן שהקירוב בנקודה $t = 1$ הוא ~ 0.2428 .

$$: x' = 1 + tx^2, x(0) = 0, t = 1, h = 0.2 \cdot 3$$

(א) מתקיים $f(t, x) = 1 + tx^2$, $t_0 = 0, x_0 = 0$. נסמן $t_i = 0.2i$ כאשר $0 \leq i \leq 5$ (כאשר $t = 1$) $t_5 = 0.2 \cdot 5 = 1$.

$$i. (t_0, x_0) = (0, 0) \Rightarrow m_0 = f(0, 0) = 1 \Rightarrow x - 0 = 1 \cdot (t - 0) \Rightarrow x = t \Rightarrow (t_1, x_1) = (0.2, 0.2) \Rightarrow$$

$$ii. m_1 = f(0.2, 0.2) = 1 + 0.2^3 = 1.008 \Rightarrow x - 0.2 = 1.008(t - 0.2) \Rightarrow x = 1.008t - 1.6 \cdot 10^{-3} \Rightarrow (t_2, x_2) = (0.4, 0.4016) \Rightarrow$$

$$iii. m_2 = f(0.4, 0.4016) \cong 1.0645 \Rightarrow x - 0.4016 = 1.0645(t - 0.4) \Rightarrow x = 1.0645t - 0.0242 \Rightarrow (t_3, x_3) = (0.6, 0.6145) \Rightarrow$$

$$iv. m_3 = f(0.6, 0.6145) \cong 1.2265 \Rightarrow x - 0.6145 = 1.2265(t - 0.6) \Rightarrow x = 1.2265t - 0.1214 \Rightarrow (t_4, x_4) = (0.8, 0.8598) \Rightarrow$$

$$v. m_4 = f(0.8, 0.8598) \cong 1.5914 \Rightarrow x - 0.8598 = 1.5914(t - 0.8) \Rightarrow x = 1.5914t - 0.4133 \Rightarrow (t_5, x_5) = (1, 1.1781) \Rightarrow$$

ומכאן שהקירוב בנקודה $t = 1$ הוא ~ 1.1781 .

קירוב Picard

$$: x' = x^2 + 3t^2 - 1, x(1) = 1 \cdot 1$$

מתקיים: $t_0 = 1, x_0 = 1, f(t, x) = x^2 + 3t^2 - 1$.

$$\varphi_0(t) = x_0 = 1$$

$$\varphi_1(t) = x_0 + \int_1^t f(s, \varphi_0(s)) ds = 1 + \int_1^t (1 + 3s^2 - 1) ds = 1 + \int_1^t 3s^2 ds = 1 + [t^3]_1^t = 1 + t^3 - 1 = t^3$$

$$\varphi_2(t) = x_0 + \int_{t_0}^t f(s, \varphi_1(s)) ds = 1 + \int_1^t (s^6 + 3s^2 - 1) ds = 1 + \left[\frac{1}{7} s^7 + s^3 - s \right]_1^t = 1 + \frac{1}{7} t^7 + t^3 - t - \frac{1}{7} - 1 + 1 = \frac{1}{7} t^7 + t^3 - t + \frac{6}{7}$$

$$: x' = x + e^{x-1}, x(0) = 1.2$$

$$.t_0 = 0, x_0 = 1, f(t, x) = x + e^{x-1} : \text{מתקיים}$$

$$\varphi_0(t) = x_0 = 1$$

$$\varphi_1(t) = 1 + \int_0^t f(s, 1) ds = 1 + \int_0^t (1 + e^0) ds = 1 + 2t$$

$$\varphi_2(t) = 1 + \int_0^t f(s, 1 + 2s) ds = 1 + \int_0^t (1 + 2s + e^{2s}) ds = 1 + t + t^2 + \frac{1}{2}e^{2t} - \frac{1}{2}e^0 = \frac{1}{2}(e^{2t} + 1) + t + t^2$$

$$: x' = 1 + t \sin x, x(\pi) = 2\pi.3$$

$$.t_0 = \pi, x_0 = 2\pi, f(t, x) = 1 + t \sin x : \text{מתקיים}$$

$$\varphi_0(t) = x_0 = 2\pi$$

$$\varphi_1(t) = 2\pi + \int_\pi^t f(s, 2\pi) ds = 2\pi + \int_\pi^t (1 + s \cdot \sin(2\pi)) ds = 2\pi + \int_\pi^t ds = 2\pi + t - \pi = t + \pi$$

$$\varphi_2(t) = 2\pi + \int_\pi^t f(s, s + \pi) ds = 2\pi + \int_\pi^t (1 + s \cdot \sin(s + \pi)) ds = 2\pi + \int_\pi^t (1 - s \cdot \sin s) ds = 2\pi + [s + s \cdot \cos s - \sin s]_\pi^t =$$

$$\int x \sin x ds = \left[\begin{array}{l} u = x \Rightarrow d' = 1 \\ v' = \sin x \Rightarrow v = -\cos x \end{array} \right] = -x \cos x - \int (-\cos x) ds = -x \cos x + \sin x$$

$$= 2\pi + t + t \cos t - \sin t - \pi - \pi \cos \pi + \sin \pi = 2\pi + t(1 + \cos t) - \sin t - \pi + \pi + 0 = 2\pi + t(1 + \cos t) - \sin t$$

פתרון מד"ר ע"י טורים סביב נקודה רגולרית:

מציאת $y(x)$ ע"י טורי חזקות:

$$x_0 = 0 \text{ פיתוח סביב } y'' = xy, y(0) = 1, y'(0) = 0.1$$

נחש פתרון: $y(x) = \sum_{k=0}^{\infty} c_k x^k$ כאשר מהנתונים נובע: $c_0 = 1, c_1 = 0$. מתקיים: $y''(x) = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2}$.

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} = x \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{k+1} \Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0 \Rightarrow$$

$$2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - c_{k-1}] x^k = 0$$

השוואת מקדמי פולינום:

$$c_2 = 0, \quad (k+2)(k+1)c_{k+2} - c_{k-1} \Rightarrow c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)} \Rightarrow \text{solution: } \begin{cases} c_{k+3} = \frac{c_k}{(k+3)(k+2)} \\ c_0 = 1, c_1 = 0, c_2 = 0 \end{cases}$$

כיוון ש- c_{k+3} תלוי ב- c_k ו- $c_1 = c_2 = 0$, נשאר עם מקדמים מהצורה c_{3k} בלבד:

$$k=0: c_3 = \frac{c_0}{2 \cdot 3} = \frac{1}{2 \cdot 3}; \quad k=3: c_6 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} \dots \Rightarrow c_{3k} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \dots \cdot (3k-1) \cdot 3k} \Rightarrow$$

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{2 \cdot 3 \cdot \dots \cdot (3k-1) \cdot 3k} x^{3k}$$

$$: y'' + xy' + y = 0, y(0) = 0, y'(0) = 1.2$$

מתקיים: $x_0 = 0$ ו- $c_1 = 1, c_0 = 0$.

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + x \sum_{k=1}^{\infty} k c_k x^{k-1} + \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k =$$

$$2c_2 + c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} + (k+1)c_k] x^k = 0 \Rightarrow 2c_2 + c_0 = 2c_2 = 0 \Rightarrow c_2 = 0;$$

$$(k+2)(k+1)c_{k+2} + (k+1)c_k = 0 \Rightarrow \begin{cases} c_{k+2} = -\frac{c_k}{k+2} \\ c_0 = 0, c_1 = 1, c_2 = 0 \end{cases} \Rightarrow c_{2k+1} \text{ נשאר עם מקדמים מהצורה } c_0 = 0 \text{ ו-} c_k \text{ תלוי רק ב-} c_k$$

$$c_1 = 1 = \frac{1}{1}; \quad k=1: c_3 = -\frac{c_1}{3} = -\frac{1}{1 \cdot 3}; \quad k=3: c_5 = -\frac{c_3}{5} = \frac{1}{1 \cdot 3 \cdot 5} \dots \Rightarrow y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!!} x^{2k+1}$$

$$: (1+x^2)y'' + 2xy' = 0, y(0) = 0, y'(0) = 1.3$$

$$.x_0 = 0, c_0 = 0, c_1 = 1 \text{ מתקיים:}$$

$$(1+x^2)y'' + 2xy' = (1+x^2) \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + 2x \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2 \sum_{k=1}^{\infty} k c_k x^k =$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2 \sum_{k=1}^{\infty} k c_k x^k = \sum_{k=0}^{\infty} [(k+2)(k+1)c_{k+2} + k(k-1)c_k + 2k c_k] x^k = 0 \Rightarrow$$

$$(k+2)(k+1)c_{k+2} + k(k-1)c_k + 2k c_k = 0 \Rightarrow \begin{cases} c_{k+2} = -\frac{k(k-1) + 2k}{(k+2)(k+1)} c_k = -\frac{k}{k+2} c_k \\ c_0 = 0, c_1 = 1 \end{cases}$$

$$: c_{2k+1} \text{ תלוי רק ב-} c_k \text{ ו-} c_0 = 0 \text{ המקדמים מהצורה}$$

$$c_1 = 1; \quad c_3 = -\frac{1}{3} c_1 = -\frac{1}{3}; \quad c_5 = -\frac{3}{5} c_3 = \frac{1}{5}; \quad c_7 = -\frac{5}{7} c_5 = -\frac{1}{7} \Rightarrow c_k = \frac{(-1)^k}{2k+1} \Rightarrow y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

$$: y'' + \frac{2}{3}y' + y = 0, y(0) = 1, y'(0) = 0.4$$

$$\text{כנראה יש טעות בשאלה. מהפתרון: } y = \frac{\sin x}{x} \text{ נובע שנוסחת הנסיגה שאמורה להתקבל היא: } c_{k+2} = -\frac{1}{(k+2)(k+3)} c_k \text{, בשונה מזו שמתקבלת מהשאלה.}$$

$$\text{פתרון מד"ר ע"י טורים סביב נקודה סינגולרית רגילה: } (x_0 = 0)$$

$$: 2xy'' + y' + xy = 0.1$$

$$P(x) = 2x, Q(x) = 1, R(x) = x \Rightarrow \lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} = \lim_{x \rightarrow 0} \frac{x}{2x} = \frac{1}{2}; \quad \lim_{x \rightarrow 0} \frac{x^2 R(x)}{P(x)} = \lim_{x \rightarrow 0} \frac{x^2}{2} = 0 \Rightarrow x_0 \text{ היא נקודה סינגולרית רגילה}$$

$$y = (x-0)^r \sum_{k=0}^{\infty} c_k (x-0)^k = \sum_{k=0}^{\infty} c_k x^{k+r} \Rightarrow y' = \sum_{k=0}^{\infty} (k+r)c_k x^{k+r-1}, y'' = \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r-2} \Rightarrow$$

$$2xy'' + y' + xy = 2x \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r-2} + \sum_{k=0}^{\infty} (k+r)c_k x^{k+r-1} + x \sum_{k=0}^{\infty} c_k x^{k+r} = \sum_{k=0}^{\infty} 2(k+r)(k+r-1)c_k x^{k+r-1} +$$

$$+ \sum_{k=0}^{\infty} (k+r)c_k x^{k+r-1} + \sum_{k=0}^{\infty} c_k x^{k+r+1} = \sum_{k=0}^{\infty} (k+r)(2k+2r-1)c_k x^{k+r-1} + \sum_{k=0}^{\infty} c_k x^{k+r+1} =$$

$$\sum_{k=-1}^{\infty} (k+r+1)(2k+2r+1)c_{k+1} x^{k+r} + \sum_{k=1}^{\infty} c_{k-1} x^{k+r} = r(2r-1)c_0 x^{r-1} + c_0 x^{r+1} +$$

$$\sum_{k=0}^{\infty} [(k+r+1)(2k+2r+1)c_{k+1} + c_{k-1}] x^{k+r} \Rightarrow F(r) = r(2r-1) \text{ (מקדם של } -x^{r-1} \text{)}$$

$$(k+r+1)(2k+2r+1)c_{k+1} + c_{k-1} = 0 \Rightarrow c_{k+1} = -\frac{c_{k-1}}{(k+r+1)(2k+2r+1)} \Rightarrow c_k = -\frac{c_{k-2}}{(k+r)(2k+2r-1)}$$

$$: 3x^2 y'' + 2xy' + x^2 y = 0.2$$

$$P(x) = 3x^2, Q(x) = 2x, R(x) = x^2 \Rightarrow \lim_{x \rightarrow 0} \frac{xQ(x)}{P(x)} = \lim_{x \rightarrow 0} \frac{2x^2}{3x^2} = \frac{2}{3}; \quad \lim_{x \rightarrow 0} \frac{x^2 R(x)}{P(x)} = \lim_{x \rightarrow 0} \frac{x^4}{3x^2} = 0 \Rightarrow x_0 \text{ נקי סינגולרית רגילה}$$

$$3x^2 y'' + 2xy' + x^2 y = 3x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r-2} + 2x \sum_{k=0}^{\infty} (k+r)c_k x^{k+r-1} + x^2 \sum_{k=0}^{\infty} c_k x^{k+r} =$$

$$\sum_{k=0}^{\infty} 3(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=0}^{\infty} 2(k+r)c_k x^{k+r} + \sum_{k=0}^{\infty} c_k x^{k+r+2} = \sum_{k=0}^{\infty} (k+r)(3k+3r-1)c_k x^{k+r} + \sum_{k=2}^{\infty} c_{k-2} x^{k+r} =$$

$$(3r^2 - r)c_0 x^r + (r+1)(3r+2)c_1 x^{r+1} + \sum_{k=2}^{\infty} [(k+r)(3k+3r-1)c_k + c_{k-2}] x^{k+r} = 0 \Rightarrow F(r) = 3r^2 - r = r(3r-1)$$

$$(k+r)(3k+3r-1)c_k + c_{k-2} = 0 \Rightarrow c_k = -\frac{c_{k-2}}{(k+r)(3k+3r-1)}$$

$$: x^2 y'' + x y' + (x-2)y = 0.3$$

$$P(x) = x^2, Q(x) = x, R(x) = x-2 \Rightarrow \lim_{x \rightarrow x_0} x \cdot \frac{x}{x^2} = 1; \lim_{x \rightarrow x_0} x^2 \cdot \frac{x-2}{x^2} = -2 \Rightarrow \text{היא נקודה סינגולרית רגילה}$$

$$x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r)c_k x^{k+r-1} + (x-2) \sum_{k=0}^{\infty} c_k x^{k+r} = \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r} +$$

$$\sum_{k=0}^{\infty} (k+r)c_k x^{k+r} + \sum_{k=0}^{\infty} c_k x^{k+r+1} - \sum_{k=0}^{\infty} 2c_k x^{k+r} = \sum_{k=0}^{\infty} [(k+r)(k+r-1)c_k + (k+r)c_k - 2c_k] x^{k+r} + \sum_{k=1}^{\infty} c_{k-1} x^{k+r} =$$

$$[r \cdot (r-1) + r - 2]c_0 x^{k+r} + \sum_{k=1}^{\infty} [((k+r)^2 - 2)c_k + c_{k-1}] x^{k+r} \Rightarrow F(r) = r^2 - r + r - 2 = r^2 - 2$$

$$((k+r)^2 - 2)c_k + c_{k-1} = 0 \Rightarrow c_k = -\frac{c_{k-1}}{(k+r)^2 - 2}$$