Numerical Analysis 1

Assignment 8

1. Let A be a matrix with eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0$ and corresponding eigenvectors u^i $(i = 1, 2, .., n)$, such that the first two eigenvector are $u^1 = (0, 1, \cdots)$, $u^2 = (1, 0, \cdots)$. Let $x^{(0)} = \sum_{i=1}^{n} u^i$. Show that

$$\lim_{k\to\infty} \frac{(A^{k+1}x^{(0)})_i}{(A^k x^{(0)})_i} = \begin{cases} \lambda_2 \ , & i = 1 \\ \lambda_1 \ , & i = 2 \end{cases}$$

2. Prove that among all polynomials satisfying $Q_{n+1}(1) = 1$ the one with minimal maximum norm on $[a, b]$ (including the case where $1 \not\in [a, b]$) is

$$Q_{n+1}(x) = \frac{T_{n+1}\left(\frac{x-\mu}{\delta}\right)}{T_{n+1}\left(\frac{1-\mu}{\delta}\right)}$$

where $\mu = \frac{a+b}{2}, \delta = \frac{a-b}{2}$ and T_{n+1} is Chebyshev polynomial.

3. The following table lists the values of $f(x) = cos(x)$ at some points with round off error $\leq \frac{1}{2}10^{-5}$:

x_i	$f(x) = cos(x)$
1.100	0.45360
1.190	0.37166
1.199	0.36329
1.200	0.36236
1.201	0.36143
1.210	0.35302
1.300	0.26750

   (a) Use these values to approximate $f'(1.2)$ with the formula $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$ with $h = 0.1$, $h = 0.01$ and $h = 0.001$.

   (b) Compare your results with $f'(1.2) = -sin(1.2) \approx -0.93204$. Explain how the magnitude of h affects the approximation error.

   (c) The bound of the approximation error is known to be

$$\frac{\epsilon}{h} + \frac{M\,h^2}{6}$$

   Where ϵ is a bound for the round off error in approximating $f(x_i)$ and M is a bound for $|f^{(3)}(x)|$. For each h from (a) find $\frac{\epsilon}{h}$, $\frac{M\,h^2}{6}$ and the total error bound. Explain their meaning.

(d) Find the optimal h for the approximation in (a) by minimizing the bound in (c) for f(x)=cos(x).

4. Using Taylor polynomials:

   (a) Derive an approximation formula for $f'(x_0)$ using the known values $f(x_0 - 2h)$, $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$.

   (b) Determine the asymptotic order of accuracy of the formula in part (a).

   (c) Derive a three-point central-difference formula for $f''(x_0)$ and determine its asymptotic accuracy.

5. What is the asymptotic accuracy of the following approximation:

$$f''(x) \approx \frac{2f(x) - 5f(x+h) + 4f(x+2h) - f(x+3h)}{h^2}$$

6. Consider following approximation:

$$f''(x) = \frac{af(x) + bf(x+h) + 114f(x+2h) - 56f(x+3h) + 11f(x+4h)}{12h^2} + O\left(h^3\right)$$

   Determine the coefficients a and b.