Numerical Analysis 1

Assignment 5

1. Consider the problem of finding a second order polynomial $P_2(x)$ that interpolates the function $f(x)$ and its derivative $f'(x)$, such that:

$$P_2(x_0) = f(x_0) \ , \qquad P_2'(x_1) = f'(x_1) \ , \qquad P_2(x_2) = f(x_2) \ ,$$

   i.e., $P_2(x)$ interpolates $f'(x)$ at $x = x_1$ rather than $f(x)$. Assuming $x_0 \neq x_2$ show that $P(x)$ exists for every $f(x)$ if and only if $x_1 \neq \dfrac{x_0 + x_2}{2}$.

2. (a) Find the polynomial $P(x)$ of degree ≤ 3 which best approximates $f(x) = x^4 + 3x^2$ in $[-1, 1]$ in the maximum norm.
   (that is, $P(x)$ is the polynomial of degree ≤ 3 for which $\max_{x\in[-1,1]} |f(x) - P(x)|$ is minimal.)

   (b) For $f(x)$ and $P(x)$ from (a), draw the error function $|E(x)| = |f(x) - P(x)|$ in $[-1, 1]$.
   Identify the zeros of $|E(x)|$. What is the maximal value that $|E(x)|$ attains? How many times is this value attained?

3. Consider interpolating the function $f(x) = sin(x)$ at the 6 Chebyshev points in $[0, \pi/4]$. Find these points and give an error bound.

4. Using fixed-point iteration solve the following equations

   (a) $x^3 - x - 1 = 0$ for the root in $[1,\ 2]$

   (b) $x - \cos x = 0$ (determine the interval yourself!)

   in the following way:

   (a) For each equation find one converging iteration function $g(x)$ and one diverging $g(x)$. Run the fixed-point iterations in Matlab up to 10^{-12} accuracy.

   (b) Calculate the value of $g'(x)$ at the fixed point r for each equation. Use these values to:

      i. Explain why you got convergence/divergence in (a) in each method.

      ii. Estimate the number of iterations needed to reach 10^{-12} relative accuracy using the error bound and the approximation $k \approx |g'(r)|$.

      iii. Compare this estimate with the actual number of iterations.