CS623 Winter 2012 \ Assignment #1

Ariel Stolerman

1)

a.

Let S, T be 2 convex sets, and let $Q = S \cap T$. Following is a proof that Q is also convex:

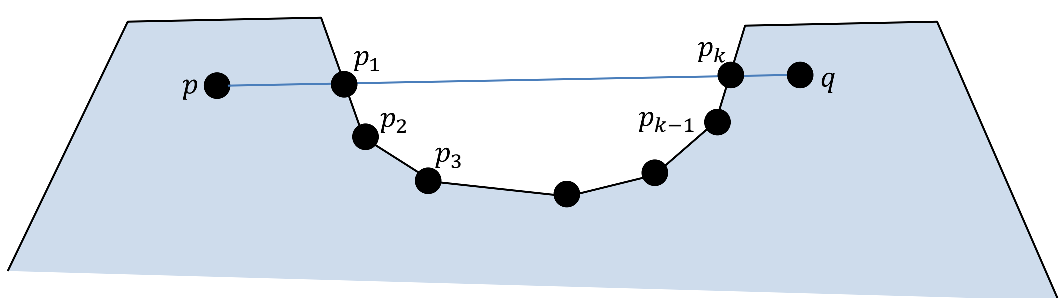
Assume $S \cap T$ is not convex, then there exist two points $p_1, p_2 \in S \cap T$ (so $p_1, p_2 \in S$ and $p_1, p_2 \in T$) such that the line (p_1, p_2) is not fully contained in $S \cap T$, and therefore is either not fully contained in S or not fully contained in T (or both). But $p_1, p_2 \in S$ and $p_1, p_2 \in T$, and since S and T are both convex, we get a contradiction.

b.

Let $\mathcal{P}$ be the smallest perimeter polygon containing the points P. Assume $\mathcal{P}$ is not convex, then there exist two points $p, q \in P$ (and in $\mathcal{P}$) such that the line (p, q) is not fully contained in $\mathcal{P}$. Therefore there exists a segment on the line (p, q), denoted $[p_1, p_k]$, such that $p_1, p_k \in \mathcal{P}$ and $(p_1, p_k) \cap \mathcal{P} = \emptyset$ (we are not guaranteed that the whole of (p, q) is not contained in $\mathcal{P}$, but certainly a sub-segment described above exists).

Since (p_1, p_k) is not contained in $\mathcal{P}$ but $p_1, p_k \in \mathcal{P}$ and are on the boundary of $\mathcal{P}$ (which is a closed set, as it is defined as a polygon), since $\mathcal{P}$ is a polygon there must exist a set of points $p_2, \dots, p_{k-1}$ on the boundary of $\mathcal{P}$ such that the segments $[p_1, p_2], \dots, [p_{k-1}, p_k] \subset \mathcal{P}$ and are all on the boundary of $\mathcal{P}$.

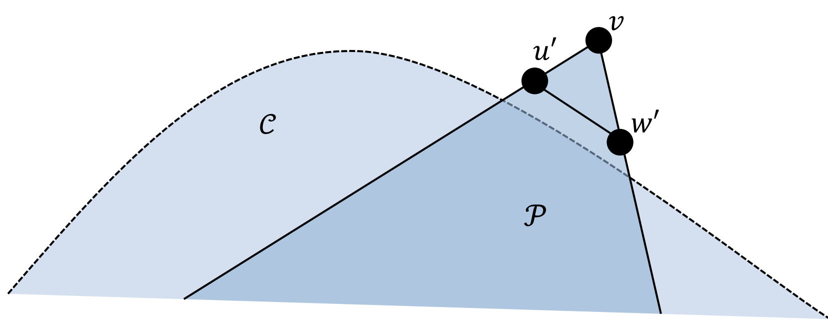
Since (p_1, p_k) is not contained in $\mathcal{P}$, that set of segments above must "go around" (p_1, p_k) in the direction towards "inside" the polygon $\mathcal{P}$, in order to satisfy that $\mathcal{P}$ will not contain (p_1, p_k) (note that it cannot go around it towards "outside" the polygon, since then $\mathcal{P}$ will contain (p_1, p_k)). But we know the length of the shortest path between any two points p_1, p_k is the length of the line between them $d(p_1, p_k)$. Therefore the length of the boundary of $\mathcal{P}$ between p_1, p_k, which is $\sum_{i=1}^{k-1} d(p_i, p_{i+1})$, is strictly larger than $d(p_1, p_k)$. Therefore we can create a new polygon $\mathcal{Q}$ that is identical to $\mathcal{P}$ with the only difference of connecting p_1, p_k directly instead of going through $p_2, \dots, p_{k-1}$, and by that we also satisfy $\mathcal{P} \subset \mathcal{Q}$. Therefore $\mathcal{Q}$ has perimeter strictly smaller than $\mathcal{P}$'s, and it contains the points P as well, which is a contradiction to the definition of $\mathcal{P}$. Therefore we have proven that the polygon with the smallest perimeter containing the points P is convex.



c.

Let $\mathcal{C}$ be a convex set containing the set of points P, and $\mathcal{P}$ the smallest perimeter polygon containing the set of points P as well. Assume that $\mathcal{P}$ is not fully contained in $\mathcal{C}$. Then since $\mathcal{P}$ is a polygon, at least one of its vertices is not contained in $\mathcal{C}$, since if all vertices of $\mathcal{P}$ were in $\mathcal{C}$, and $\mathcal{C}$ is convex, then all the edges of $\mathcal{P}$ would have also been contained in $\mathcal{C}$, thus $\mathcal{P}$ would have been completely contained in $\mathcal{C}$ is contradiction to the assumption.

Therefore there exists a vertex of $\mathcal{P}$, denoted v, such that $v \notin \mathcal{C}$, and since $\mathcal{C}$ contains all the points in P, v cannot be in P. But now we can construct a polygon Q that will contain all points in P and will have a smaller perimeter than $\mathcal{P}$: let u, w be vertices of $\mathcal{P}$ such that there are edges in $\mathcal{P}$: $[u, v], [v, w]$. We can take 2 points in the environment of v, denoted u', w', such that $u' \in [u, v], w' \in [v, w]$ and the triangle (u', v', w') doesn't contain any points in P (since $v \notin P$ there exists such environment). Then we construct Q exactly like $\mathcal{P}$ only not taking the triangle (u', v, w'), and taking the segment $[u', w']$ instead. By the triangle inequality, we shortened the perimeter since $d(u', w') < d(u', v) + d(v, w')$. Therefore we got a contradiction to the definition of $\mathcal{P}$ as the smallest perimeter polygon containing the set of points P.



2)

Let E be the unsorted set of n segments that are the edges of a convex polygon. Following is a $O(n \lg n)$ algorithm that computes from E a list containing all vertices of the polygon, sorted in clockwise order:

Let P be the set of $2n$ points, which construct the segments in E.

1. Sort the point by their x-value and secondary sort by y-value (such that all points are sorted by x-value, and within each set with the same x-value, the points are sorted by their y-value).
2. Remove all points in even locations (remove duplicate points).
3. Run Graham's scan, substituting the construction of the sorted point set with the phases above, and return its output.

<u>Correctness</u>:

Phases 1 and 2 make sure we get the sorted set of n points (as n segments have together $2n$ endpoints, with duplicates), and the correctness of the rest is derived from the correctness of Graham's scan.

<u>Running time</u>:

The total running time is the same as Graham's scan, $O(n \lg n)$, as we substituted the original sorting with a different one.

3)

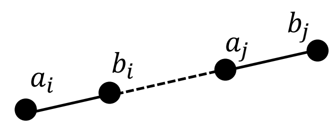
a.

Let S be a set of n line segments in the plane. Following is a proof that the convex hull of S is exactly the same as the convex hull of the $2n$ endpoints of the segments:

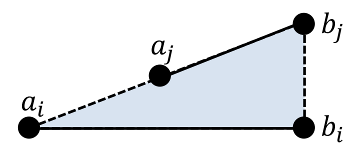
Denote the set of line segments $S = \{[a_1, b_1], \dots, [a_n, b_n]\}$ and the set of their endpoints $T = \{a_1, \dots, a_n, b_1, \dots, b_n\}$.

First, let C be the convex hull of the set of points T, then $a_i, b_i \in C$ for all $1 \le i \le n$ and also $(a_i, b_i) \subset C$ for all $1 \le i \le n$, since C is a convex set, therefore $[a_i, b_i] \subset C$ for all $1 \le i \le n$. Let $[a_i, b_i], [a_j, b_j]$ be any two of the line segments. Since C is the convex hull with respect to T then $[a_i, b_i], [a_j, b_j], [a_i, a_j], [b_i, b_j], [a_i, b_j], [a_j, b_i] \subset C$. Next we look at all possible relations between $[a_i, b_i]$ and $[a_j, b_j]$:

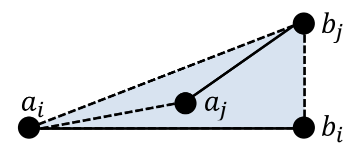
- If a_i, b_i, a_j, b_j are on the same line, and assuming without loss of generality that a_i, b_j are the two endpoints that are farthest apart, then C must include the segment $[a_i, b_j]$, which is the same constraint that is imposed on the convex hull of the segments $[a_i, b_i], [a_j, b_j]$.



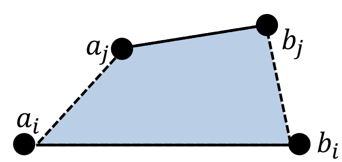
- If a_i, b_i, a_j, b_j create a triangle (e.g. a_i, a_j, b_j are on the same line), then this triangle is contained in C. But this triangle also contains all segments $[s, t]$ such that $s \in [a_i, b_i], t \in [a_j, b_j]$ (this can be checked mathematically by checking that for each $p \in [s, t]$ the angles $\angle p a_i b_j, \angle p b_j b_i, \angle p b_i a_i$ are $\le 180°$, i.e. p is contained in the triangle; the same for the next cases).



- If all points create a concave quadrilateral, and let a_j be the concaving point, then since C is a convex set, it contains the triangle that contains this concave quadrilateral, and again it contains all segments $[s, t]$ as described above.



- If all points create a convex quadrilateral, then it is definitely also a convex set with respect to the line segments $[a_i, b_i], [a_j, b_j]$.



Therefore the convex hull C is a convex set with respect to S as well. Since C is the convex hull of T, there exists no smaller convex set that will include all points in T, therefore there exists no smaller convex set that will include all line segments in S, so C is the convex hull of S.

If C is the convex hull of S then for any line segment $[a_i, b_i]$, $a_i, b_i \in C$ and $(a_i, b_i) \subset C$, i.e. C is a convex set with respect to T. Due to the uniqueness of the convex hull and what we showed before (about the convex hull of T being the convex hull of S), C must be the convex hull of T.

b.

Let $\mathcal{P}$ be a non-convex polygon on n vertices. Following is a $O(n)$-time algorithm that computes the convex hull of $\mathcal{P}$:

Let $p_1, \dots, p_n$ be the n vertices of $\mathcal{P}$. I'll assume we have the sequence of points in order of their appearance on the perimeter of the polygon $\mathcal{P}$ (say, clockwise starting at p_1) otherwise it is impossible to achieve $O(n)$ (as sorting will be required). We will follow Graham's scan algorithm, but substitute the sorting of $p_1, \dots, p_n$ by their x-coordinate with the following (given the list of points in order on the polygon's perimeter):

1. Find the point p_i with minimum x-value. As shown in class, if more than one such point exists, we can "tilt" the x-axis such that all points will have a unique x-value.
2. In the same manner, find the point p_j with maximum x-value.
3. Let $p_i, \dots, p_j$ be the subsequence of the points in order for the upper polygon, and the reverse order of $p_{j+1}, \dots, p_n, p_1, \dots, p_{i-1}$ (i.e. $p_{i-1}, \dots, p_1, p_n, \dots, p_{j+1}$) the subsequence for the lower polygon.

Now we simply continue the same as in Graham's scan algorithm.

Correctness:

We iterate over the vertices of the polygon in order of appearance (clockwise), and since it is a concave polygon, the upper part of the chain (points $p_i, \dots, p_j$) always "wraps" the lower part such that the lower part can't "break in" the upper part. I.e. if we build the upper part of the convex hull using the vertices $p_i, \dots, p_j$, there will not be any point out of $p_{j+1}, \dots, p_n, p_1, \dots, p_{i-1}$ that either below the line that goes through p_i, p_j or contained in that upper convex hull. Symmetrically this applies to the lower convex hull, and so the complete convex set that is constructed in a similar way to Graham's scan is the convex hull of the given concave polygon.

Running time:

Given the sequence of points by their order on the perimeter of $\mathcal{P}$, finding the subsequences for the upper and lower polygons (steps 1-3) is linear. The rest of the steps are identical to Graham's scan, which are linear (only the sorting at the beginning of the original algorithm is super-linear). Thus the algorithm is $O(n)$ in total.

4)

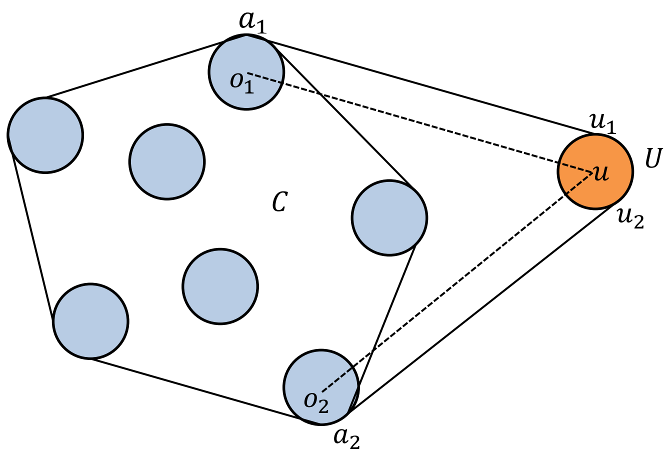
a.

Proof by induction over the number of unit circles in the S, n.

If $n = 1$, there is a single circle which is the convex hull itself (and smaller form will not contain the entire sole circle), which satisfies the boundary condition.

Assume any convex hull boundary for $n = 2, \dots, m$ is formed of straight line segments and pieces of circles in S, and denote that convex hull C. Now we add a new unit circle U. If $U \subset C$, then the new convex hull for the $m + 1$ circles is simply C, which satisfies the boundary condition by the induction assumption. Otherwise, we will construct the new convex hull for the whole $m + 1$ unit circles as follows:

Pick some point p such that $p \notin U, p \notin C$ and $[p, o] \cap C = \emptyset$, where $o \in U$ is the center of the circle U. Now find the points o_1, o_2 out of all m circle center points of the circles in C with the smallest and largest angles $\angle poo_1, \angle poo_2$. Let the two circles with centers o_1, o_2 be $O_1, O_2 \subset C$, respectively. Find the line segments $[a_1, u_1]$ and $[a_2, u_2]$ such that a_1 is on the perimeter of O_1, u_1 is on the perimeter of U and the segment $[a_1, u_1]$ is parallel to $[o_1, u]$, and in a similar fashion $[a_2, u_2]$ is defined. The closed shape we got, that is defined by the perimeter: $u_1 \xrightarrow[\text{perimeter of } U]{\text{over the}} u_2 \to a_2 \xrightarrow[\text{perimeter of } c]{\text{over the}} a_1 \to u_1$ is the new convex hull for all $m+1$ unit circles, where $u_1 \to u_2$ is a part of the circle U, $u_2 \to a_2$ and $a_1 \to u_1$ are straight line segments and $a_2 \to a_1$ is on the original perimeter of the convex hull of the m circles, thus by assumption is constructed of only straight line segments and parts of circles. Therefore the new convex hull is also constructed as required.



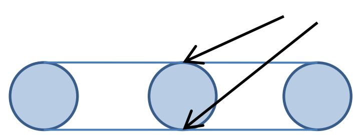
Correctness for the convex hull construction above: denote the new set C'. To show it is convex, it is sufficient to show that for any pair of circles O, U where U is the newly added circle and O is one of the m original circles in C, $\forall a \in O$, $u \in U: [a, u] \subset C'$. But in the way we selected $[a_1, u_1], [a_2, u_2]$ each "tube" from any O to U is fully contained in C'. Moreover, if these segments were note selected to be parallel:

- If they were towards "inside" C' we would have had a concave shape.
- If they were towards "outside" C', we could have narrowed the shape in area up to the point of tangency.

Therefore C' is the convex hull of the $m+1$ given circles.

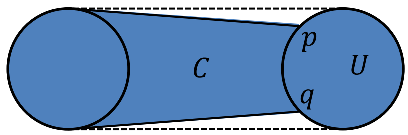
b.

This in fact is not correct, as in the following example, where the 2 tangent points of the middle circle appear on the boundary of the convex hull:



But assuming the meaning is that no circle can appear twice on the boundary of the convex hull, where each appearance includes more than one point, following is a proof:

If S includes only one unit circle U then the convex hull $C = U$ itself, therefore U appears only once on the boundary of C (the entire perimeter of U). If S includes more than one circle, assume that it appears twice on the boundary of C such that each appearance is more than one point. Let the points p, q, r, s be the points on the circle U such that from these points the boundary of C continues towards away from U. Assume that the points p, q are on a sector of U that is inside C (i.e. the sector p, q on U is not on the boundary of C), then lines go out from p, q to continue the boundary of C. At the best case, there is only one additional circle at that direction, such that the lines that go out from p, q away from U to that circle are tangent to that circle. But by the definition of p, q, the sector (p, q) is shorter than the sector between the tangent points on the other circle, and so it concludes that C will not be convex, a contradiction:



Thus any circle can appear only once on the boundary of C.

c.

Let C be the convex hull of S and C' the convex hull of S'. First assume the circle $U \in S$ appears on the boundary of C, and let u be its center. Assume that u is not on the boundary of C', and look at all pairs of vertices of C', denoted o_1, o_2. By definition, u is contained in the intersection of the convex sets defined by all lines $o_1 o_2$. Then for every o_1, o_2 we can "pump" o_1, o_2, u to the unit circles, then they all cover the area around them distant by 1. But since we pumped o_1, o_2 and u by the same measurement, then the new boundary is a line tangent to the circles formed around o_1, o_2 and the circle around u, U, does not intersect that line, hence it is still contained in the convex set generated by the tangent above. Therefore U is not on the boundary of C, in contradiction to the assumption, so u must be on the boundary of C'.

Now assume u is a vertex of C', and assume U is not on the boundary of C. In a similar fashion to the previous part, this means U is contained by the intersection of all convex sets defined by the lines tangent to pairs of circles on the boundary of C. Then we can "pump down" the circles to their center, and u will be fully contained by pumped-down shape (which is now a polygon), therefore will not be on its boundary, in contradiction to the assumption. Therefore U must be on the boundary of C.

d.

Given what is previously proven, to find the convex hull of a set of unit circles S, denoted C, we do as follows:

- If S contains 1 circle, return it; if it contains 2 circles, return the tube formed by crossing 2 tangent lines between them.
- Otherwise, find the convex hull of the set center points of the circles in S, using Graham's scan, denoted C'.
- For any 3 consecutive vertices of C', denoted p, q, r, add to the boundary of C:
  - The line segment $[p', q']$, which is parallel to $[p, q]$, distant by 1 from it and forms a rectangle $pp'q'q$ towards outside of C'.
  - The line segment $[q'', r']$, defined in a similar fashion with respect to q, r.
  - The circle sector $q' \to q''$ of the unit circle with the center point q, which goes "outside" the shape we're building (i.e. if s is the intersection point of the lines $p'q', q''r'$, then the sector contained in the triangle $q'sq''$).

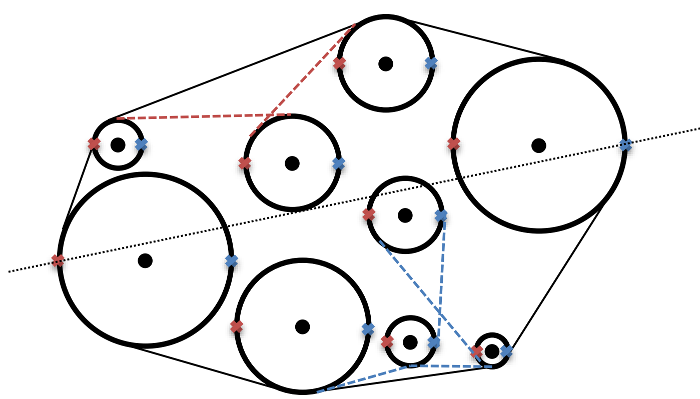
  Note: the boundary can be sufficiently defined by the finite set of points p', q', q'', r' (derived from all C-vertices triplets) and a mapping of each part to either a line segment (like $p' \to q', q'' \to r'$) or a unit-circle sector (like $q' \to q''$).
- Return C.

This algorithm clearly "pumps" the convex set it creates for the circles center points, as required. The first part of running Graham's scan is $O(n\lg n)$, and the rest is linear, concluding to $O(n\lg n)$ in total, as required.

e.

Following is a $O(n\lg n)$ algorithm (similar to Graham's scan) to find the convex hull given a set S of n circles defined by their center point and radius (o_i, r_i):

1. Find all "left" x-values of the circles, defined by $o_i.x - r_i$, and sort the points in increasing order with respect to that value, in the sequence denoted X_L. Denote the points $a_i = (o_i.x - r_i, o_i.y)$.
2. Find all "right" x-values of the circles, defined by $o_i.x + r_i$, and sort the points in decreasing order with respect to that value, in the sequence denoted X_R. Denote the points $b_i = (o_i.x + r_i, o_i.y)$.
3. Cross a line between a_i with the minimum x and b_i with the maximum x, and set:
   - $Upper$ – the subsequence of points o_i from X_L above that line.
   - $Lower$ – the subsequence of points o_i from X_R below that line.
4. Continue as in Graham's scan algorithm, with the following differences:
   - If one circle contains the other completely, take the larger one and continue from it.
   - The upper points are trailed in left-order (i.e. the order of a_i) and the lower points – right-order.
   - Instead of crossing lines between consecutive points o_i, o_{i+1}, find the 2 tangents between the 2 corresponding circles (found in constant time given $o_i, r_i o_{i+1}, r_{i+1}$; ignore the other 2 tangents that cross the line $o_i o_{i+1}$) and out of those 2 tangents pick the one that starts at the point of tangency that you first encounter trailing clockwise on the circumference of the circle of o_i from the last tangency point (of the tangent chosen between o_{i-1}, o_i). For the first tangent, start trailing from the left-most a_i.
   - The angle that will be checked for orientation violation will be the one between the tangent line chosen at the current phase and the one chosen at the previous phase.
5. Return the sequence of tangency points of the selected tangents, starting at the first (meaning: if we enumerate these points t_i starting at 1, then each $t_{2i-1}t_{2i}$ is a line segment and each $t_{2i}t_{2i+1}$ is a circle circumference sector), with the list of corresponding boundary circles O_i with center o_i (such that the sector $t_{2i}t_{2i+1}$ is on the circumference of the circle O_i).



Correctness:

The correctness derives from the correctness of Graham's scan, where taking the tangent lines instead of the lines between the circle centers handles the fact the objects contained are circles of varying radii.

Running time:

All sorting phases take $O(n \lg n)$. All other phases that run in linear time in Graham's scan, are linear here as well, since calculating the tangent lines and checking the orientation is done in constant time, for each iteration. That concludes to a total of $O(n \lg n)$ time.