Theorem:

Let S be a set, then there is no onto mapping $g: S \to P(S)$ (from S to its power set).

Remarks:

- A mapping $f: S \to T$ is onto if $\forall t \in T \exists s \in S: f(s) = t$. f is also called "surjective".
- A mapping $f: S \to T$ is one-to-one if $\forall x, y \in S: x \neq y \Rightarrow f(x) = f(y)$. f is also called "injective".
- A mapping $f: S \to T$ is a correspondence if f is one-to-one and onto. f is also called "bijective".

Proof:

Assume that $g: S \to P(S)$ is onto. Define a set $G = \{s \in S \mid s \notin g(s)\} \subset S$. In particular, $G \in P(S)$. Since g is onto, there exists $t \in S$ such that $g(t) = G$. Then, if $t \in G$, then $t \in g(t)$ then $t \notin G$ by the definition of G, a contradiction. If $t \notin G$ then $t \notin g(t)$ then by the definition of G: $t \in G$, again a contradiction.

Therefore there cannot exist an onto-mapping g from a set to its power set. □

When do 2 sets have the same cardinality:

For finite sets it is immediate. For infinite sets, there are infinitely sizes of infinity. The intuition to distinguish between different infinities, is to look at how we count: when we count, we construct a correspondence between a finite set of consecutive integers and those objects that we count.

It comes in handy for infinite sets:

Theorem:

There are as many even natural numbers as there are natural numbers: $|\mathbb{N}| = |\mathbb{N}_{even}|$.

Proof:

We will construct a correspondence: $f: \mathbb{N} \to \mathbb{N}_{even}$, that is $\forall n \in \mathbb{N}: f(n) = 2n$.

f is one-to-one: let $m, n \in \mathbb{N}$ then if $n \neq m$ then $f(n) = 2n \neq 2m = f(m)$, as required.

f is onto: let m be an even integer, then $m = 2n$ and for $n \in \mathbb{N}$: $f(n) = m$.

Visualization:

$\mathbb{N}$	$\mathbb{N}_{even}$
1	2
2	4
3	6
…	…
n	$2n$

Theorem:

There is a correspondence between $\mathbb{N}$ and $\mathbb{Z}$:

Proof:

$g: \mathbb{Z} \to \mathbb{N}$ is defined as: $z \to \begin{cases} 2z, & z \geq 0 \\ -2z - 1, & z < 0 \end{cases}$ – map the positive to even naturals, and negative to odd naturals.

For instance: $1 \to 2, 2 \to 4, -1 \to 1, -2 \to 3$ and so on.

To continue the prof we need to formally show that g is one-to-one and onto.

Some cardinality arithmetic:

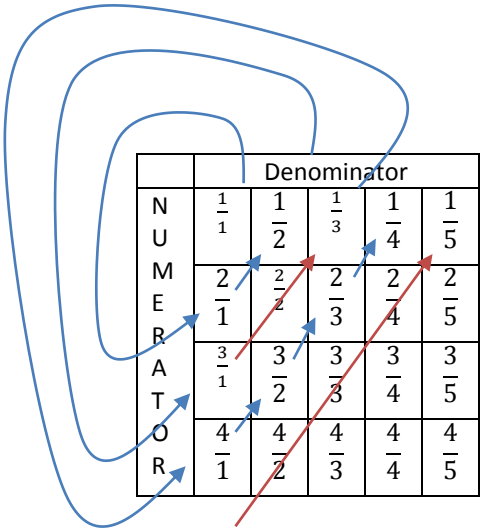
$|S| \leq |T|$ if there is an injection (one-to-one mapping) from S to T.

Theorem:

There is a correspondence between $\mathbb{N}$ and $\mathbb{Q}$.

Proof:

First, we need to show that there are as many positive rational numbers as there are naturals, as we've seen a correspondence $\mathbb{Z} \to \mathbb{N}$ and we can apply that logic here. So we'll look for a correspondence in $\mathbb{N} \to \mathbb{Q}^+$. The book writes in a table all rational numbers in their numerator/denominator representation, and maps them to $\mathbb{N}$ diagonally, skipping over numbers already seen:



The blue arrows show the direction of the mapping, the red ones jump over those already mapped (e.g. $\frac{1}{1}$ is mapped, then we skip over $\frac{2}{2}, \frac{3}{3}$ and so on).

Recounting the Rationals (Calkin and Wilf, 2000):

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \dots$$

If we look at $(b_n) = 1,1,2,1,3,2,3,1,4,3,5, \dots$

b_n is the number of hyperbinary representations on n (representations as a sum of powers of 2, where each power repeats 0,1, or 2 times in the representation). For instance, $b_5 = 2$ means that 5 has 2 binary representations (sums of powers of 2): $4 + 1$ or $2 + 2 + 1$.

The paper actually shows that the sequence of numerators is b_n.The paper shows that the sequence holds: $f(2n + 2) = f(n) + f(n + 1)$. For instance, $f(10) = f(5) + f(4) = 2 + 3 = 5$:

$4 = 1 \cdot 2^2 = 2 \cdot 2^1 = 2 \cdot 2^0 + 1 \cdot 2^1 - 3$ representations; and we have already seen 5 has 2.

In addition: $f(2n + 1) = f(n)$. These 2 formulas give us a recursion that allows us computing all hyperbinary representations.

We can consider these numbers as a tree: If the parent is of the form $\frac{i}{j}$ then the left child is $\frac{i}{j+i}$ and the right child is $\frac{i+j}{j}$.

The paper states that the numerator and denominator of the parent, left and right child are relatively prime. If there's a rational in the tree $\frac{r}{s}$ such that r, s are NOT relatively prime ($\gcd(r,s) > 1$), then let $\frac{r}{s}$ be on the minimal level. If $\frac{r}{s}$ is a left child, then its parent is $\frac{r}{s-r}$. But since r, s are not relatively prime, also $r, s-r$ are not, so $r, s-r$ would have a common divisor – a contradiction to the minimality of the level of $\frac{r}{s}$. If $\frac{r}{s}$ is a right child, its parent is $\frac{r-s}{s}$, and again a contradiction.

The rest is in the paper.

Theorem:

There is no correspondence between $\mathbb{N}$ and $\mathbb{R}$.

Proof:

By contradiction, assume there is such correspondence $g: \mathbb{N} \to \mathbb{R}$. Look at the decimal expansions of $g(1), g(2), g(3), \ldots$:

	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$g(1)$	**0**	1	5	5
$g(2)$	4	**7**	8	4
$g(3)$	7	9	**9**	0
...				
$g(k)$				

$g(k)$ is defined to have at each ith position a number that is different than the ith position of $g(i)$. So we define that number $a = 0.a_1a_2a_3a_4 \ldots$ such that $\forall i \geq 0: a_i \neq g(i)$. But a is one of the numbers, i.e. for some j: $g(j) = a$, but then a has to have a_j different than the jth digit of $g(j) = a$, i.e. a_j, a contradiction.

Therefore there is no correspondence between $\mathbb{N}$ and $\mathbb{R}$. □

The Halting Problem

We show that $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a Turing machine and } M \text{ accepts } w\}$ is undecidable.

Note: A_{TM} is trivially Turing-recognizable: simulate M on w, if it accepts – accept; otherwise – either reject or loop forever.

Given a finite Σ, there is a correspondence between $\mathbb{N} \to \Sigma^*$. Therefore we can enumerate all encodings of a Turing machine and inputs – we go over all strings lexicographically and check if they are valid; same goes for input: it has an enumeration.

So if A_{TM} was decidable, we could have check if a machine halts. If so – simulate it and return whether it accepts or rejects, and if it doesn't halt – reject. Therefore we can construct the EXACT opposite machine, and that machine will appear as an input at some index j – and then the machine at position j, with a given input word of itself, it will do what it doesn't do – a contradiction.

So: there are Turing machines $M_1, M_2, M_3, \ldots$ and input words $w_1, w_2, w_3, \ldots$. Consider the table:

	M_1	M_2	M_3	…	M_j
M_1	**acc**	rej	rej		…
M_2	acc	**rej**	acc		…
M_3	rej	acc	**acc**		…
…					
M_j	**rej**	**acc**	**rej**		**???**

There are also languages that are not even Turing-recognizable. Consider a lexicographic enumeration of all possible inputs over $\{0,1\}^*$, and the characteristic vector of any possible TM such that at any position i is has 0 if it rejects w_i and 1 otherwise. We will continue next week.