

## Exercises

1. Prove that NL is closed under union and under intersection.<sup>1</sup>
2. (a) Show that PAL, the language of all palindromes over  $\{0,1\}$ , can be decided using logarithmic space. What is the running time of your Turing machine?  
(b) Show that there is a non-deterministic Turing machine that decides  $\overline{\text{PAL}}$  in linear time and logarithmic space.
3. Consider the problem CYCLE-1 defined as follows:  
Input: A directed graph with out-degree at most one.  
Question: Does it contain a cycle?  
Show that CYCLE-1  $\in$  L.
4. In the problem UPATH the input is an *undirected* graph  $G = (V, E)$ , and two vertices  $s, t \in V$ , and the question is "Is  $s$  connected to  $t$  in  $G$ ?"  
(a) Show that UPATH  $\in$  NL.  
Remark: UPATH is known to be in L and is complete for a class called SL.  
(b) Show that  $\overline{2\text{-Col}} \in$  NL.
5. Show that deciding whether a directed graph contains an odd (directed) cycle is NL-complete.
6. Let us define the class  $\text{NL}^*$  just like NL, except that we allow the head of the witness tape to move in both directions. In more detail,  $\text{NL}^*$  is the class of all languages  $L$  for which there exists a Turing machine with the following criteria:  
Input tape: Read only, move in both directions.  
Witness tape: Read only, move in both directions.  
Work tape: Read-Write, move in both directions.  
The machine itself is deterministic (the guesses are the value of the witness tape). The space complexity is the size of the work tape, and is bounded by  $O(\log n)$ . We say the machine accepts an input if and only if there exists a setting for the witness tape, with which the machine accepts.  
Prove that  $\text{NP} \subseteq \text{NL}^*$ , and conclude that if  $\text{P} \neq \text{NP}$  then  $\text{NL} \neq \text{NL}^*$ .

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<sup>1</sup>That is, if  $A, B \in \text{NL}$  then  $A \cup B \in \text{NL}$  and  $A \cap B \in \text{NL}$ .