Homework 4
Due 21/04/2009

1. Prove or disprove or show that the statement is equivalent to an open question:
   (a) $\mathsf{DTIME}(2^n) \subsetneq \mathsf{NTIME}(2^{2n})$.
   (b) $\mathsf{P} \neq \mathsf{NP}$ or $\mathsf{NP} \neq \mathsf{EXP}$.
   (c) There exists a $k > 0$ such that $\mathsf{NP} \subseteq \mathsf{DTIME}(n^k)$.
   (d) For any language $L \in \mathsf{NP} \cap \mathsf{coNP}$, $\mathsf{NP}^L = \mathsf{NP}$.

2. (a) Let $\Sigma_2\mathsf{SAT}$ denote the following decision problem: given a quantified formula ψ of the form $\psi = \exists x_1, \ldots, x_n \; \forall y_1, \ldots, y_n. \; \phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$, where ϕ is a CNF formula, decide whether ψ is true. Prove that if $\mathsf{P} = \mathsf{NP}$ then $\Sigma_2\mathsf{SAT} \in \mathsf{P}$.
   (b) For $k_1, k_2 \in \mathbb{N}$ define the problem (k_1, k_2)-Coloring Extension (in short, (k_1, k_2)-CE) as follows: given a graph G and a set of vertices S, decide whether any k_1-coloring of S can be extended to a k_2-coloring of G. Show that $(2,3)\text{-CE} \in \Pi_2^p$ and that $(2,2)\text{-CE} \in \mathsf{coNP}$.

3. (a) Prove that if $\mathsf{NTIME}(n) \subseteq \mathsf{DTIME}(n^{10})$ then $\mathsf{P} = \mathsf{NP}$.
   Hint: First use a padding argument to show that for any $k \geq 1$, $\mathsf{NTIME}(n^k) \subseteq \mathsf{DTIME}(n^{10k})$.
   (b) Prove that if every unary NP-language is in P then $\mathsf{EXP} = \mathsf{NEXP}$, and conclude that if $\mathsf{EXP} \neq \mathsf{NEXP}$ then there exists a language $L \in \mathsf{NP} \setminus \mathsf{P}$ that is not NP-complete.
   Remark: It is known that there exists a language $L \in \mathsf{NP} \setminus \mathsf{P}$ that is not NP-complete assuming the weaker assumption $\mathsf{P} \neq \mathsf{NP}$ (Ladner's Theorem).

4. We define the class $\mathbf{S}_2^p$ as the set of all languages L for which there exist a polynomial-time Turing machine M and a polynomial p such that for all $x \in \{0,1\}^*$,

$$x \in L \Rightarrow \exists y \in \{0,1\}^{p(|x|)} \; \forall z \in \{0,1\}^{p(|x|)}. \; M(x,y,z) = 1$$
$$x \notin L \Rightarrow \exists z \in \{0,1\}^{p(|x|)} \; \forall y \in \{0,1\}^{p(|x|)}. \; M(x,y,z) = 0$$

   (a) Is $\mathbf{S}_2^p$ closed under complement?
   (b) Prove that $\mathbf{S}_2^p \subseteq \Sigma_2^p \cap \Pi_2^p$.