Homework 1
Due 17/03/2009

1. Let $f, g : \mathbb{N} \to \mathbb{N}$ be two functions. Recall that $f = O(g)$ if there exists a $c > 0$ such that $f(n) \leq c \cdot g(n)$ for every sufficiently large n. We say that $f = \Omega(g)$ if $g = O(f)$ and that $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$. Also, we say that $f = o(g)$ if for any $\varepsilon > 0$, $f(n) \leq \varepsilon \cdot g(n)$ for every sufficiently large n. Finally, we say that $f = \omega(g)$ if $g = o(f)$.

   Prove or disprove:

   (a) $(5n)! = O(n!^5)$.

   (b) If $f(n) = O(n)$ then $10^{f(n)} = O(2^n)$.

   (c) $\log(n!) = \Theta(n \log n)$.

   (d) Every two functions f, g satisfy $f = O(g)$ or $g = O(f)$.

   (e) There exists a function f such that $f(n) = O(n^{1+\varepsilon})$ for any $\varepsilon > 0$ but $f(n) = \omega(n)$.

2. For two languages L_1, L_2 define $L_1 \Delta L_2 = (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$. We say that a class C is closed under Δ if $L_1, L_2 \in C$ implies $L_1 \Delta L_2 \in C$. For each class decide if it is closed under Δ (or show that it is equivalent to an open question): P, NP, $\mathsf{NP} \cap \mathsf{coNP}$.

3. Prove that each of the following problems can be solved by a polynomial time algorithm:

   (a) Input: A graph G and a positive integer k.
   Question: Does G contain a vertex of degree at least $\log_2 |V(G)|$ or a clique of size k? ($V(G)$ denotes the vertex set of G).

   (b) Input: A list of n positive integer numbers $A_1, \ldots, A_n$ and a number T. All the numbers are given in unary representation (i.e., a number k is represented as 1^k).
   Question: Does exist a subset $S \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in S} A_i = T$?

   (c) Input: A $3CNF$ formula ϕ in which each clause contains exactly 3 distinct literals and each variable occurs exactly 3 times.
   Question: Is ϕ satisfiable?
   Hint: Use the fact that any regular bipartite graph has a perfect matching.[1]

4. Let $A \subseteq \{0,1\}^*$ be a language which satisfies $|A \cap \{0,1\}^n| = n^3$ for all $n \geq 10$. Prove that $A \in \mathsf{NP}$ implies $A \in \mathsf{coNP}$.

[1] A regular graph is a graph where each vertex has the same number of neighbors. A matching in a graph is a set of edges without common vertices. A perfect matching is a matching which matches all vertices of the graph.